Chapter 1
Probability, Percent, Rational Number Equivalence

Traditionally, seventh grade starts by gathering up everything students have learned about numbers and arithmetic, in a way that increases their flexibility with operations while illuminating the underlying algebraic structure of the number system. Experience suggests that this traditional approach does not work; students, weak as well as strong, find this an uninteresting review of either what they already know or what they are unlikely to learn through repetition. For this reason, we approach this review with a new topic, probability, with the idea that its extrinsic interest will attract the students' attention, while exhibiting the importance of arithmetic operations in context. Another reason for starting the year with probability activities is to develop a culture of thinking about mathematics as a way to investigate real world situations. A third reason is that activities at the beginning of the year can help foster a classroom culture of discussion and collaboration.

Throughout this chapter students are provided with opportunities to review and build, based on knowledge from previous grades, fluency with fractions, percents, and decimals and recognize equivalent forms of rational numbers. Students should understand that fractions, percents and decimals are all relative to a whole. Students will also compare and order fractions (both positive and negative.).This chapter concludes with a section specifically about solving percent and fraction problems, including those involving discounts, interest, taxes, tips, and percent increase or decrease. As students model mathematics, they begin to apply properties of operations (the "field axioms") informally, leading to the formalization in chapter 3.

This chapter is only an introduction to probability; students will work more with probability in Chapter 7. Additionally, the concepts studied in 7th grade around chance processes (theoretical and experimental probabilities) lay a foundation for work in later years when students study conditional probability, compound events, evaluate outcomes of decisions, use probabilities to make fair decisions, etc.

This is students' first formal introduction to probability. In the first section students will study chance processes, experiments or situations for which they know the possible outcomes but do not know which outcome will occur at any run of the experiment. Students will look at probabilities as ratios expressed as fractions, decimals, or percents (part:whole). In previous coursework, students worked with the concept of ratio, but in 7th grade, they will be led to a precise definition. Thus students should be familiar with the idea of part:whole, part:part, and whole:whole relationships. It is important to emphasize that in this chapter only part:whole relationships are discussed. In Chapter 4 we will be talking about part:part relationships where distinguishing between these relations is going to be important. In addition, it should become clear that we can (depending upon where we want to have the emphasis) take either a part:whole or a part:part relationship and convert it to the other. For example, if 3/5 of the class are girls we know that 2/5 are boys and the ratio of girls to boys is 3:2. Later, in chapter 7, students will discuss "odds" which are part:part relationships.

Probabilities will be determined from the results or outcomes of experiments. They will learn that the set of all possible outcomes for an experiment is a *sample space*. They will recognize that the probability of any single *event* (a subset of the sample space) can be expressed in terms of impossible, unlikely, equally likely, likely, or certain or as a number between 0 and 1, inclusive. Students will focus on two concepts in the probability of an event: *experimental* (*empirical*) and *theoretical*. They will understand the commonalities and differences between experimental and theoretical probability in given situations. This will conclude the first section.

While studying probability, students continue their study of rational numbers. They will convert rational numbers to decimals and percents and will look at their placement on the number line. This lays the foundation for 8th grade where students study irrational numbers to complete the real number system. Hence, in the next section, students solidify and practice rational number sense through the careful *review* of fractions, decimals and percents. The two key objectives of the second section are a) students should confidently articulate relationships among equivalent fractions, decimals, and percents using words, models, and symbols and b) students should understand and use models to find portions of different wholes.

The concept of equivalent fractions naturally leads students to the issues of ordering and estimation. Ordering positive and negative fractions will be connected to the number line. It is important that students develop estimation skills in conjunction with both ordering and operating on positive and negative rational numbers. Lastly, students look at percent as being a fraction with a denominator of 100. Percent and fraction contexts in this section will be approached intuitively with models.

The chapter concludes with a section in which students continue to solve contextual problems with fractions, decimals and percent but begin to transition from relying solely on models to writing numeric expressions. In subsequent chapters students will extend their understanding by writing equations and proportional equations using variables.

Section 1.1. Investigate Chance Processes. Develop/Use Probability Models

The mathematics emphasized in this chapter reflects the importance in today's society of being able to understand basic concepts of probability. References to probability are all around us,including weather forecasting. Suppose you have some outdoor plans made for a particular day and the weather report says that the chance of rain is 70%. Should you still go ahead with your plans or should you cancel them for another day? Another example of probabilities in daily life comes from the world of sports. A batting average involves calculating the probability of a player hitting the ball. That is, a batting average is a statistic (hits ÷ atbats) that is developed from past history. However, it is used in a theoretical sense: a batter with a .300 average is 50% more likely to get on base as a batter with a .200 average. So let's say your favorite baseball player is batting .300. This means that when he or she goes up to the plate, there is only a 30% chance of getting a hit! Playing the lottery is another instance of probability in real life. Millions of people around the world spend their money on lottery tickets in hopes of winning the big jackpot and becoming millionaires. But do these people realize how low their chances of winning actually are?

Probability is a vehicle for students to engage in a new mathematical topic while reviewing and practicing whole number and rational number arithmetic. We are also preparing the way for the study of statistical inference (Chapter 7), given that probability provides a mathematical description of randomness, such as the chance variation observed in the outcomes of randomized experiments and random samples. This development occurs as students consider and discuss with their peers the outcomes of a variety of probabilistic situations.

The mathematical study of probability dates to the 15th century and is based on problems involving gambling. Most historians think that it originated in an unfinished dice game. The French mathematician Blaise Pascal received a letter from his friend Chevalier de Méré, a professional gambler, who attempted to make money gambling with dice. Chevalier de Méré's predicament involved two games of dice. In the first one, he made money by betting that he could roll a 6 on at least one of four consecutive rolls of a die. Empirical experience led him to believe that he would win more times than he would lose. He reasoned correctly that the chance of getting a six in one roll of a die is $\frac{1}{6}$. He then incorrectly thought that in four rolls of a die, the chance of getting one six would be $\frac{4}{6} = \frac{2}{3}$. Though his reasoning was faulty, he made considerable money over the years in making this bet. Today we know that the probability of winning this bet is $1 - \left(\frac{5}{6}\right)^4$, or 51.8%.

When folks would no longer bet on this proposition, de Méré modified the game by betting even money (original bet is either doubled or lost) that double 6's would turn up at least once in 24 throws of a pair of fair dice. This seemed like a good bet, but he began losing money. He reasoned correctly that the chance of getting a double six in rolling a pair of dice is $\frac{1}{36}$ However, he erred in thinking that in 24 rolls of a pair of dice, the chance of getting one double six would be $\frac{24}{36} = \frac{2}{3}$.

Why would the first game be profitable for de Méré, even though his reasoning was faulty, and the second game not?

In the first game, in a roll of a die, there are six possible outcomes: $1, 2, 3, 4, 5, 6$. If the die is fair, the probability of getting a six is $\frac{1}{6}$. Likewise, the probability of getting no six in one roll of a fair die is $\frac{5}{6}$.

The probability of getting no six in four rolls is:

$$P(\text{no six in four rolls}) = \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \left(\frac{5}{6}\right)^4$$

Thus in four rolls of a fair die, the probability of getting at least one six is:

$$\begin{aligned} P(\text{at least one six in four rolls}) &= 1 - P(\text{no six in four rolls}) \\ &= 1 - \left(\frac{5}{6}\right)^4 \\ &= 0.517747 \end{aligned}$$

Thus the probability of getting at least one six in four rolls of a fair die is 0.517747. Out of 100 games, de Méré would on average win 52 games. Out of 1000 games, he would on average win 518 games.

In the second game, in a roll of a pair of dice, there are a total of 36 possible outcomes (the six outcomes of the first die combined with each of the six outcomes of the second die). Out of these 36 outcomes, only one of them is a double six. So, the probability of getting a double six is $\frac{1}{36}$ in rolling a pair of dice. Likewise, the probability of not getting a double six is $\frac{35}{36}$. The probability of getting no double six in 24 rolls of a pair of dice is:

$$\begin{aligned} P(\text{at least one double six in 24 rolls}) &= 1 - P(\text{no double six in 24 rolls}) \\ &= 1 - \left(\frac{35}{36}\right)^4 \\ &= 0.4914 \end{aligned}$$

Thus the probability of getting at least one double six in 24 rolls of a pair of fair dice is 0.4914. On average, de Méré would only win about 49 games out of 100 and his opponent would win about 51 games out of 100 games.

Based on empirical data (he lost a lot of money), he knew something was not quite right in the second game of dice. So he challenged his renowned friend Blaise Pascal to help him find an explanation. Pascal shared the problem with Pierre Fermat and together they solved the problem, which is often marked as the beginning of the era of the mathematical theory of probability.

Probability is about how likely a particular event is, and measures the chance that itwill occur. In probability, we study chance processes, which concern experiments or situations where we know which outcomes are possible.

An *experiment* is an activity whose results can be observed and recorded. Each of the possible results of an experiment is an *outcome*. If we toss a fair coin (*i.e. heads and tails are equally likely to occur therefore it's a fair coin*), there are two distinct possible outcomes: head (H) and tails (T).

The set of all possible outcomes for an experiment is a *sample space*. The sample space S for rolling a fair die is $S = \{1, 2, 3, 4, 5, 6\}$. An event is a collection of outcomes, a set in the sample space. The set of all even-numbered rolls $\{2, 4, 6\}$ is a subset of all possible rolls of a die $\{1, 2, 3, 4, 5, 6\}$ and is an event.

So, how do you measure the chance of an event? Well, that depends on the event itself which falls into two different categories.

Let's consider dice used in Las Vegas casinos. Licensed dice manufacturer companies in the US are inspected for quality control and the dice are calibrated to within 1/10,000 of an inch - weighted and balanced to perfection, so that one side is just as heavy as any of the other five sides. You'll often see the boxman put a die between his/her thumb and middle finger to give it a little spin, or a rudimentary check for fairness.

As demonstrated above, probability surrounds us in our daily lives, whether it be in playing at the Casino, dabbling in the stock market, gauging weather for a clime of Mt. Everest, or planning a picnic. Casino operators have capitalized on the probabilities of gambling, creating a "house advantage" that is beaten only by the luck of a few.

Let's introduce theoretical probability with this question:

What is the probability of rolling a six with a fair die?

With a roll of a fair die, there are six possible outcomes that are equally likely $1, 2, 3, 4, 5, 6$, so each has a probability of $\frac{1}{6}$. Hence, the theoretical probability of rolling a six with a fair die is $\frac{1}{6}$. When the probability of an event is known, or at least can be determined through an analysis of the situation, the probability of an event, in an experiment where all outcomes are equally likely, can be expressed as follows:

$$\frac{\text{Number of Outcomes in the Event}}{\text{Number of Possible Outcomes}}$$

Going back to our scenario, in 1967, the Sands casino in Vegas, recently upgraded with a Teamster's loan, was shut down for using crooked dice. It was discovered by a pit boss, who had suspicions about some dice players and got security to let him into their hotel room, where they confiscated a satchel full of crooked dice already logo'ed and ready to slip into games at a dozen Vegas casinos.

The second category includes events whose theoretical probability is not known, and can only be established through empirical data or based on observed data generated from past experiments or data collection. Let's consider a new question.

Is the die we are about to use a fair die?

As with the boxman we could examine the die or spin it to see if it has a balanced spin, as described above. weighed, measured, spun to check for fairness (as alluded to above). Alternatively, an experiment could be performed to see if there is any significant deviation from the theoretical expectation described above. So, a die is rolled a number of times and the outcomes are recorded. (In general, data gathered from an experiment are referred to as *empirical* data. Then we calculate the experimental probability as:

$$\frac{\text{Number of Observed Occurrences of the Event}}{\text{Total Number of Trials}}$$

Although it would be impossible to conduct an infinite number of trials, we can consider the long-run relative frequency as a closer approximation to the actual probability or the theoretical probability as the size of the data set (sample) increases. This is referred to as the *law of large numbers*. It is also known as Bernoull's theorem, in honor of Jakob Bernoulli (1654-1705). How many trials are enough? It depends upon how sure you want to be that the die is fair: if you want 80% confidence, you will have to roll the die a lot more than if you are satisfied with 60% confidence.

Approximate the probability of a chance event by collecting data on the chance process that produces it and observing its long-run relative frequency, and predict the approximate relative frequency given the probability. For example, when rolling a number cube 600 times, predict that a 3 or 6 would be rolled roughly 200 times, but probably not exactly 200 times. 7.SP.6

An investigation regarding the law of large numbers was conducted by John Kerrich during World War II. In April of 1940, while visiting family in Copenhagen, John Kerrich was caught in the Nazi invasion and was imprisoned. To pass time, Kerrich tossed a coin 10,000 times. On his release Kerrich published an account of his experiments in a short book entitled *An Experimental Introduction to the Theory of Probability*. A sample of his results is in Table 1.1. The relative frequency column on the right is obtained by dividing the number of heads by the number of tosses of the coin.

Table 1.1

Number of tosses	Number of Heads	Relative Frequency
10	4	0.400
50	25	0.500
100	44	0.440
500	255	0.510
1,000	502	0.502
5,000	4,034	0.504
10,000	5,067	0.507

As the number of tosses increased, Kerrich obtained heads close to half the time. The long-run relative frequency for Kerrich's tosses gives a result of 5,067/10,000, or approximately $\frac{1}{2}$

As seen in the Sands Casino scenario, some probabilities cannot be determined by the analysis of possible outcomes of an event; instead, they can only be determined through gathering empirical data. Why? Because the outcomes are not equally likely and/or a large number of trials will need to be conducted to predict the approximate relative frequency that will be a close approximation of the theoretical probability. For example, conducting an experiment to see how often a Hershey's Kiss or a thumb-tack will land on its base.

In Chapter 7 we will use statistics to gain information about a population by examining a sample of the population. Generalizations about a population from a sample are valid only if the sample is representative of that population, and a sample of 1000 people provides more reliable and convincing data about the larger population than does a survey of 5 people. Subsequently the larger the number of trials (people surveyed), the more confident you can be that the data reflect the larger population.

Understand that the probability of a chance event is a number between 0 and 1 that expresses the likelihood of the event occurring. Larger numbers indicate greater likelihood. A probability near 0 indicates an unlikely event, a *probability around 1/2 indicates an event that is neither unlikely nor likely, and a probability near 1 indicates a likely event. 7SP5*

Previously we made a statement that probability is about how likely an event is. Let's consider the following example.

Example 1.

A Vegas fair die is tossed. Let $S = 1, 2, 3, 4, 5, 6$. Let's calculate each of the following probabilities.

a. the event A that the outcome is a 5;

b. the event B that the outcome is an even number;

c. the event C that the outcome is a number greater than 20;

d. the event D that the outcome is a number less than 20.

Solution. The expression $P(X)$ represents the probability (or likelihood or chance) that event X will occur. You can quantify probability with a fraction, percent, or decimal number. Given, each of the 6 numbers in set S has an equal chance of being rolled,

a. If we replace X by A, the outcome is 5, then $P(A) = \frac{1}{6}$.

b. If we replace X by $B = \{2, 4, 6\}$, then $P(B) = \frac{3}{6} = \frac{1}{2}$.

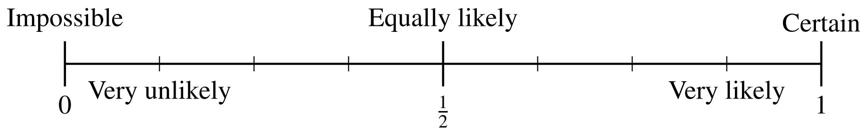
Why is $P(B) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6}$, but in de Mérés case this calculation failed? We will explore this further in the next few pages.

c. If we replace X by C, the event C is impossible. The set C has no members (this is called the *empty set*, denoted by $C = \varnothing$), then $P(C) = \frac{0}{6} = 0$.

d. If we replace X by D, the event D is certain to occur. D is the whole sample space (denoted by $D = S$) so we say the chance of D happening is 1, that is $D = S$, so $P(D) = \frac{6}{6} = 1$

To clarify, in Example 1(**c**), event C is the empty set. An event such as C that has no outcomes is an *impossible event* and has a probability of 0. In Example 1(**d**), event D consists of rolling a number less than 20. Because every number in S is less than 20, the $P(D) = 1$. An event that has a probability 1 is a *certain event*.

The number line is an important visual model of how likely an event can be.



In summation, the likelihood that an event will occur is expressed by a number called the *probability* of an event, where the probability ranges from 0 (impossibility) to 1 (certainty), or equivalently, when they are given as percentages, between 0% and 100%. A probability of 0% means that the event cannot possibly occur, as seen in Example 1(**c**). A probability of 100% means that the event is certain to occur, as seen in Example 1(d), and the greater the probability, the more likely an event is to occur. For example, if we flip a coin, the probability that it will land up is 0.5 or 50%, since we expect, either on the basis of the coin's symmetry or by data gathered from past experience, that half the time we obtain heads and half the time tails.

A probability of 50% means the event is as likely to occur as not to occur. In particular, we note:

If A is any event and S is the sample space, then $0 \leq P(A) \leq 1$.

The discussion with regards to a fair die being tossed, resulted in distinct possible outcomes of a chance process that were all equally likely. If each possible outcome is equally likely, then we call a probability model for such a process a *uniform probability model*. If there are n possible outcomes, the probability of each outcome is $\frac{1}{n}$.

Consider again Example 1. A toss of the die results in any of the sides equally likely to land face up. So the probability of rolling a 1 is equal to the probability of rolling a 2, which is equal to the probability of rolling a 3, and so on. That is,

$$P(\text{roll a } 1) = P(\text{roll a } 2) = P(\text{roll a } 3) = P(\text{roll a } 4) = P(\text{roll a } 5) = P(\text{roll a } 6).$$

Mathematically this means that given the sample space $S = \{x_1, x_2, \ldots, x_k\}$, and outcomes that are equally likely, then $P(x_1) = P(x_2) = \ldots = P(x_k)$, where $P(x)$ represents the probability of outcome x.

Additionally, notice

$$P(\text{roll a } 1) = \tfrac{1}{6},\ P(\text{roll a } 2) = \tfrac{1}{6},\ P(\text{roll a } 3) = \tfrac{1}{6},\ P(\text{roll a } 4) = \tfrac{1}{6},\ P(\text{roll a } 5) = \tfrac{1}{6},\ P(\text{roll a } 6) = \tfrac{1}{6}.$$

When you roll a fair die, the probability of rolling a 1 or a 2 or a 3 or a 4 or a 5 or a 6 (Example 1(**d**) the result is 1. Therefore,

$$P(\text{roll a } 1) + P(\text{roll a } 2) + P(\text{roll a } 3) + P(\text{roll a } 4) + P(\text{roll a } 5) + P(\text{roll a } 6) = 1,$$

hence

$$P(x_1) + P(x_2) + \ldots + P(x_k) = 1.$$

Example 2.

Let's consider additional questions relating to Example 1. A fair die is tossed. Let $S = \{1, 2, 3, 4, 5, 6\}$. Calculate each of the following probabilities. .

a. the event that the outcome is a 1, 2, 3, 4, or 6;

b. the event that the outcome is a 5.

Solution.

a. Since the outcomes 1, 2, 3, 4, or 6 are distinct possible outcomes, then the probability of rolling a 1, 2, 3, 4, or 6 is the sum of the probabilities of rolling a 1, of rolling a 2, of rolling a 3, of rolling a 4, and of rolling a 6. That is, $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{5}{6}$, so the $P(1, 2, 3, 4, \text{ or } 6) = \frac{5}{6}$

b. $P(5) = \frac{1}{6}$

Yet, is there another way of looking at the probability of rolling a 5, given the information gathered in finding the probability of rolling a 1, 2, 3, 4,or 6? In Example 2(**b**), let's rephrase the question.

Calculate the probability of the event that the outcome is not a 5.

Given that the P(roll a 1) + P(roll a 2) + P(roll a 3) + P(roll a 4) + P(roll a 5) + P(roll a 6) = 1, then $P(1, 2, 3, 4, \text{ or } 6) + P(5) = 1$, which results in the $P(5) = 1 - P(1, 2, 3, 4, \text{ or } 6)$. Are we at all surprised at the outcome? No, because we've calculated $P(5)$ before. More importanty, we should have used this logic to calculate the probability that the outcome is a 1,2,3,4,6. This strategy, computing the probability of an event happening by instead computing the event it won't happen, and subtracting that from 1, is often the easiest way to proceed. Let us summarize the principles behind this.

Two events are *mutually exclusive* if they have no outcomes in common. For instance, in Example 1, event A (rolling a 5) and event B (rolling an even number: 2, 4, 6) are mutually exclusive events because no outcome is in common to both events. If we say that event F is rolling a number less than 5 (namely 1, 2, 3, 4), then events B and F would not be mutually exclusive because they have one (or more) outcomes in common (rolling a 2 or 4).

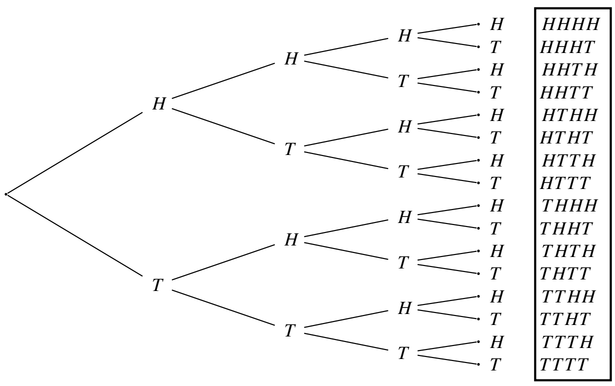
Two events, X and Y, are complementary if: a). they are mutually exclusive and b). together they make up the entire sample space.. Said another way, X and Y are complementary if everything in the sample space not in X is in Y. Because of this, the complement of X is referred to as "*not X*." Therefore,

$$P(X) + P(not\ X) = 1 \quad \text{so} \quad P(not\ X) = 1 - P(X) \quad \text{and} \quad P(X) = 1 - P(not\ X)\ .$$

Nonetheless, the example shows that counting the complementary event is especially effective for counting where an "at least" condition must be satisfied, although in Chapter 1 students will be looking for a pattern or structure to emerge.

Example 3.

Flip a coin four times. What is the probability of getting at least one head? A *tree diagram* and an *organized list* would look something like this:



For each of the two outcomes for the first flip, the tossing of the coin a second time has two possible outcomes (because the outcome of flipping the coin a second time does not depend on the outcome of flipping the coin the first time), and so on. So all together, there are $2 \times 2 \times 2 \times 2 = 2^4 = 16$ ways in which the tosses may occur. There are many ways in which one or more heads can occur. However, the complementary event is when no heads occur. There is just one way to get no head, namely, if all 4 tosses are tails. Thus, there are $16 - 1 = 15$ ways in which at least one head appears in the four tosses. Therefore, the probability of getting at least one head is $\frac{15}{16}$.

Example 4.

Flip a coin four times. What is the probability of getting heads for all four flips? One approach is to think in terms of a sequence of flips, where as soon as you get tails, you stop.

1. The probability of coming up heads on the first flip is 1/2. If you get tails on the first flip, you might as well stop, because you cannot possibly get four heads. So, half the time you stop, and half the time you keep going.
2. Assuming we kept going, then we flip the coin a second time. Again, the probability of heads is 1/2. Again, we only keep going if it comes up heads. So half the time we keep going. Overall, the chance that we will keep going is 1/2 of 1/2, or 1/4.
3. By now, 3/4 of the time we will have stopped, and 1/4 of the time we will have moved on to flip the coin a third time. Again, the probability of heads is 1/2. So, the probability that we will keep going is 1/2 of 1/4, or 1/8.
4. Finally, we have the fourth coin flip. We only get to this point 1/8 of the time. Again, the probability of heads is 1/2. The probability of four heads is 1/2 of 1/8, or 1/16.

As a shortcut, we could say that the probability of getting heads on any one throw is 1/2. The probability of getting four heads in a row therefore is (1/2)(1/2)(1/2)(1/2), or $(1/2)^4$. This example engages students in conjecturing about the multiplication rule of probability, an intuitively plausible, but more advanced probability technique developed in high school. Additionally, it is a better method, for it is less tedious than drawing a tree diagram. In particular, using example 2, the probability of getting at least one "tails" in four flips is 15/16 (because "tails" and "heads" are interchangeably probable). So, we can conclude directly from example 2, that the probability of *not* getting one "tails," which is the same as getting four heads is 1/16.

In the activity presented, the four-event experiment (or *multistage experiment*) requires four actions to determine an outcome. This is an example of a compound event: when a particular experiment is executed two or more times. In this case there is a question to consider. Does the occurrence of the event in one stage have an effect on the occurrence of the event in the other? An important factor in calculating probabilities for multistage experiments is the distinction between *independent* and *dependent* outcomes at a stage. If you flipped a coin 10 times in a row and all 10 flips came up heads, would you think that your next flip is more likely to be a tail because a tail is "due"? Saying "a tail is due", or "just one more go, my luck is due" is called The Gambler's Fallacy. Some people think "it is overdue for a tail", but a coin does not "know" it came up heads before; hence the next toss of the coin is totally independent of any previous tosses. In fact, on your next flip you are just as likely to get a tail as a head. There is nothing in the flipping history of the coin that can influence the next flip. The outcome of one coin flip is independent of the outcome of any other coin flip. In other words, the occurrence or nonoccurrence of one event has no effect on the other. In contrast, a dependent event is a second event whose result depends on the result of a first event, such as taking out a marble from a bag containing some marbles and not replacing it, and then taking out a second marble.

Example 5.

This illustration of "dependence" is that of extracting marbles from a bag. Suppose that a bag contains 16 marbles, 9 of which are green and 7 are red. The experiment is that of extracting a marble from the bag. We can see (taking the quotient of favorable outcomes to possible outcomes) that the probability that the extracted marble is green is 9/16, and that it is red is 7/16. Now, suppose the experiment is repeated a second time. The probabilities now depend upon whether or not the marble extracted the first time is replaced. If it is, the probabilities of the color of the second marble are the same as the first. But if the marble is not replaced, the probability changes, depending upon the color of the first extracted marble. If it was green, then there are 15 marbles in the bag, 8 green and 7 red; and if it was red, there are 9 greens and 6 reds. This is an issue that will be explored further in high school; in 7th grade, students should see and understand the difference.

We can use the organized list and tree diagram to calculate other probabilities as well. For example, what is the probability of getting *exactly* three heads and one tail when we toss the coin four times? Three heads and one tail occur in 4 of the 16 outcomes - namely, $HHHT, HHTH, HTHH$, and $THHH$. So the probability that either one of these four outcomes will occur is

$$\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{4}{16} = \frac{1}{4}$$

Determining the probability of the outcome of getting exactly three heads and one tail allows students to explore and conjecture about the addition rule of probability as they continue to refine their understanding.

Notice that the probability of getting *all* four heads versus the probability of getting *exactly* three heads and one tail versus the probability of getting at least one head represents an unfair game in which the events are not equally likely. All sixteen outcomes are equally likely, however the probability of getting *at least one* head versus getting *exactly* 3 heads and one tail reveals that for three heads, $HHHT$ is different from $HHTH$ because order matters. If we were to flip four coins at once, then $HHHT$ is the same as $HHTH$. This distinction will be made clear in secondary mathematics.

In order for students to determine how likely an event is, it is important that all possible outcomes are generated – for example, by modeling, using tools such as organized lists, tables, histograms or tree diagrams. This builds a strong foundation for the more advanced probability techniques that will develop in 7th grade and beyond.

Section 1.2. Equivalence and Conversion in Rational Number Forms (fraction, decimal, percent).

In this section students solidify and practice rational number sense through the careful *review* of fractions, decimals and percent. The two key objectives of this section are a) students should be confidently able to articulate with words, models and symbols, relationship among equivalent fractions, decimals, and percent and b) students should understand and use models to find portions of different wholes.

The concept of equivalent fractions naturally leads students to the issues of ordering and estimation, and ordering positive and negative fractions will be connected to the number line. It is important that students develop estimation skills in conjunction with both ordering and operating on positive and negative rational numbers.

Lastly, students look at percent as being a fraction with a denominator of 100. Percent and fraction contexts in this section will be approached intuitively with models. In Section 1.3 students will begin to transition to writing numeric expressions.

In Grade 6 students learn that the number line expands to the left of zero, exploring integers and negative fractions and decimals. They learn how to place them on the number line and how to compare numbers. They get a sense, in terms of the number line, of the interpretation of the fraction p/q as the adjunction of p copies of the line segment of length $1/q$. In Grade 7 the students learn how to represent arithmetic operations on the number line, and understand that the rules of arithmetic as they know them, extend to the system of rational numbers.

In 8th grade students explore irrational numbers, where they begin to appreciate the completeness of the real-number system. Seventh grade, to a great extent, is retrospective, that is, we focus on the structure of the mathematics already learned and how the (pictorial/concrete) models reveal how quantities are related.

The word fraction comes from the Latin "fractio" or "fractus" meaning "to break" or "broken." The use of fractions began with human observations of nature to express quantities that were less than a whole unit, such as divisions of the day. As early as 2000 BC., the Babylonians used fractions; however, the denominators of their fractions were always powers of 60 to correspond with their base system and were closely connected to their alphabet.

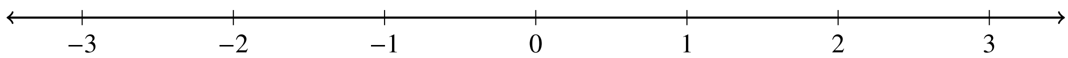
Evidence of the early stages in the development of fractions can be found in Egyptian mathematics in the Rhind Mathematical Papyrus and were very important to the Egyptians. Out of 87 problems on the Rhind, only 6 did not involve fractions.

Fraction ideas appear to have been used in many cultures.Our method of writing fractions can be attributed to the Hindus, most notably the Indian mathematician Aryabhata. By the year 1000 CE, Arabs had introduced the use of the fraction bar in their writings. Fibonacci as the first European mathematician to use the fraction bar as it is used today.

In early grades students study whole numbers as corresponding to points on the number line. Interpreting numbers as points on the number line allows for fractions to be seen as measurements with new units, created by partitioning the whole number unit into equal pieces. Recall that a fraction is a point on the number line represented by the quotient of a whole number by a counting number; a *rational number* is then a point on the number line represented by a quotient of an integer by a counting number. Therefore, integers and then the rational numbers are associated to points in the line.

How do we represent the rational number system by points on a line? With a straight edge, draw a horizontal line. Given any two points a and b on the line, we say that $a < b$ if a is to the left of b. The piece of the line between a and b is called the interval between a and b. It is important to notice that for two different points a and b we must have either $a < b$ or $b < a$. Also, recall that if $a < b$ we may also write that as $b > a$.

Pick a point on a horizontal line, mark it and call it the origin, denoted by 0. Now place a ruler with its left end at 0. Pick another point (this may be the 1 cm or 1in point on the ruler) to the right of 0 and denote it as 1. We also say that the length of the interval between 0 and 1 is one unit. Mark the same distance to the right of 1, and designate that endpoint as 2. Continuing on in this way we can associate to each positive number a point on the line. Now mark off a succession of equally spaced points on the line that lie to the left of 0 and denote them consecutively as $-1, -2, -3, \ldots$ In this way we can imagine all integers placed on the line.

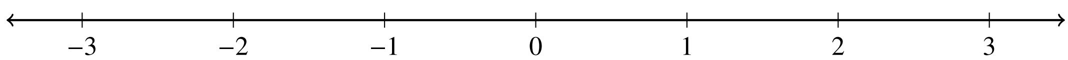


We can associate a half integer to the midpoint of any interval; so that the midpoint of the interval between 3 and 4 is 3.5, and the midpoint of the interval between −7 and −6 is −6.5. If we divide the unit interval into three parts, then the first part is a length corresponding to 1/3, the first and second parts correspond to 2/3, and indeed, for any integer p, by putting p copies end to end on the real line (on the right of the origin if $p > 0$, and on the left if $p < 0$), we get to the length representing $p/3$. We can replace 3 by any positive integer q, by constructing a length which is one qth of the unit interval. In this way we can identify every rational number p/q with a point on the horizontal line, to the left of the origin if p/q is negative, and to the right if positive.

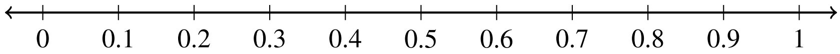
Note that, for any q, Ii p is a multiple of q, say $p = nq$, (with n an integer), then the point corresponding to the fraction p/q is precisely the integer point n. In fact, one should observe that if p/q and r/s are equivalent fractions, then the points corresponding to p/q and r/s are the same point.

Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s ore eventually repeats. 7.NS.2d

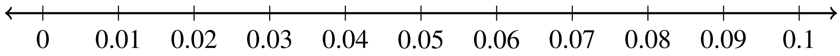
Decimals are special fractions, those with denominator 10, 100, 1000, etc. We think about decimals as "filling in" the locations on the number line between the whole numbers. We can think of plotting decimals on the number line in successive stages. At the first stage, the whole numbers are placed on a number line so that consecutive whole numbers are one unit apart.



At the second stage the decimals that have entries in the tenths place, are spaced equally between the whole numbers, breaking each interval between consecutive whole numbers into 10 smaller intervals each one-tenth unit long.



And so on.



We can think of the stages as continuing indefinitely. The digits in a decimal are like an address. When we read a decimal from left to right, we get more and more detailed information about where the decimal is located on the number line. These finer and finer partitions constitute a sort of address system for numbers on the number line: 0.543 is, first, in the neighborhood between 0.5 and 0.6, then in part of that neighborhood between 0.54 and 0.55, then exactly at 0.543. It's very similar to specifying a geographic location by giving the country, state, county, zip code, street, and street number.

Now what points on the line are represented by terminating decimals?

Rational numbers that have finite decimal expansions (terminating decimals) can be found using long division; that is, it can be expressed as a fraction whose denominator is a base 10 unit (10, 100, 1000, etc.).

$$\frac{a}{b} = \frac{n}{10} \text{ or } \frac{n}{100} \text{ or } \frac{n}{1000} \text{ or} \ldots$$

for some whole number n.

In this case then,

$$\frac{10a}{b} = n \text{ or } \frac{100a}{b} = n \text{ or } \frac{1000a}{b} = n \text{ and so on} \ldots$$

We can find the whole number n by dividing b successively into $10a, 100a, 1000a$, and so on until there is no remainder and the process terminates (at some point there is no interval left over). The point is represented by a terminating decimal.

How do we find the decimal expansion of a rational number p/q?

Example 6.

Express $\frac{7}{16}$ as a decimal using long division.

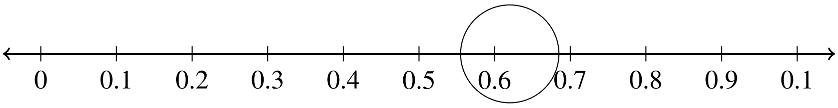
$$\begin{array}{r} 4 \\ 16\overline{)70} \\ \underline{64} \\ 6 \end{array} \qquad \begin{array}{r} 43 \\ 16\overline{)700} \\ \underline{640} \\ 60 \\ \underline{48} \\ 12 \end{array} \qquad \begin{array}{r} 437 \\ 16\overline{)7000} \\ \underline{6400} \\ 600 \\ \underline{480} \\ 120 \\ \underline{112} \\ 8 \end{array} \qquad \begin{array}{r} 4375 \\ 16\overline{)70000} \\ \underline{64000} \\ 6000 \\ \underline{4800} \\ 1200 \\ \underline{1120} \\ 80 \\ \underline{80} \\ 0 \end{array}$$

As illustrated; $70,000 = 16 \times 4375$, which means that $\frac{7}{16} = \frac{4375}{10000} = .04375$.

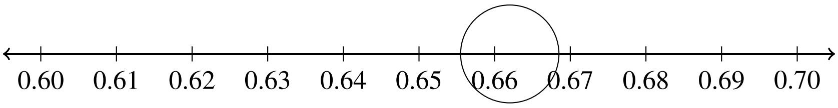
The finite decimals are the rational numbers that eventually come to fall exactly on one of the tick marks in this decimal address system. But now, this doesn't always work.

EXAMPLE 7.

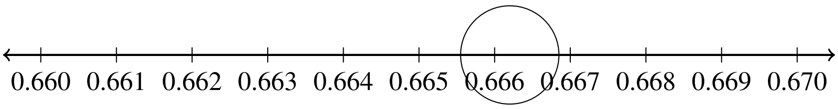
$\frac{2}{3}$ is always sitting two-thirds of the way along the third subdivision.



and then

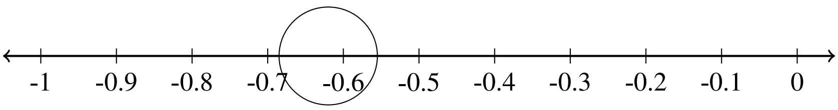


and so on...

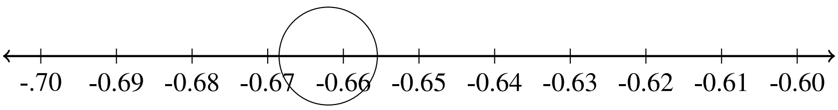


It is 0.66 plus two-thirds of a thousandth, and 0.666 plus two-thirds of ten-thousandth, and so on. The decimals 0.6, 0.66, 0.666 are successfully closer and closer approximations to $\frac{2}{3}$, that is, we say $\frac{2}{3}$ has an infinite decimal expansion consisting of entirely 6's in which we place a bar over the 6 to indicate that it repeats indefinitely: $0.\overline{6}$.

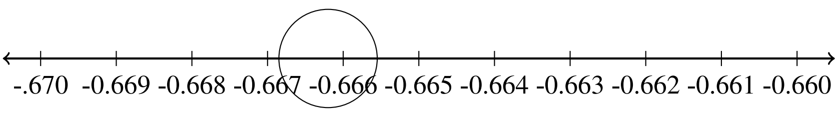
With the case of negative rational numbers the progression would be as follows; $-\frac{2}{3}$ is always sitting two-thirds of the way along the third subdivision. It is -0.66 minus two-third of a thousandth, and -0.666 minus two-third of ten-thousandth, and so on. The decimals -0.66, -0.666, -0.6666 are successfully closer and closer approximations to $-\frac{2}{3}$, that is, we say $-\frac{2}{3}$ has an infinite decimal expansion consisting of entirely 6's in which we place a bar over the 6 to indicate that it repeats indefinitely : $\frac{2}{3} = 0.\overline{6}$.



and then...



and finally...



How do we know if the fraction will terminate or repeat?

First let us look at terminating decimals. A terminating decimal, like .275, is a sum of fractions where each denominator is a power of ten (prime factors 2's and 5's). So,

$$.275 = \frac{2}{10} + \frac{7}{100} + \frac{5}{1000} = \frac{2}{10} + \frac{7}{10^2} + \frac{5}{10^3}.$$

Now, if we put these terms over a common denominator, we get

$$\frac{2(10)^2 + 7(10) + 5}{10^3} = \frac{275}{10^3}$$

In the same way, .67321 becomes

$$\frac{67321}{10^5},$$

In general a terminating decimal leads to a fraction of the form $A/10^e$ where A is an integer and e is a positive integer. So, if a decimal terminates after e terms, it can be written as a fraction with denominator 10^e. Now, since $10 = 2 \cdot 5$, we can write this as $2^e \cdot 5^e$, and we conclude that a terminating decimal can be written as a fraction whose denominator has prime factors only 2 and 5. On the other hand, we see that any fraction whose denominator (in simplest form) is a product of 2's and 5's has a terminating decimal. Let $A/(2^a 5^b)$ be such a fraction (A is an integer). Suppose $a = b$. Then our fraction is

$$\frac{A}{2^a \cdot 5^a} = \frac{A}{10^a}$$

so the decimal terminates at the ath place. If on the other hand $a \neq b$, one is larger than the other; suppose $a > b$. Then we have

$$\frac{A}{2^a \cdot 5^a} = \frac{5^{a-b}}{5^{a-b}} \frac{A}{2^a \cdot 5^a} = \frac{5^{a-b}A}{2^a 5^a} = \frac{5^{a-b}A}{10^a},$$

so terminates after the ath place. We can use the same argument if $b > a$ to show that the decimal terminates at the bth place.

Given the content in the core in grades K - 7, students will not have the background to fully understand this concept. Yet students have had experience with prime factors in Grade 6 and can intuitively understand that when the denominator (in simplest form contains only prime factors of 2's and/or 5's, the decimal will terminate. Integer exponents (simplifying expressions involving exponents) will be addressed later in Grade 8.

To convert $\frac{a}{b}$ (where a and b are integers) to a decimal we calculate $a \div b$ by long division. At each step in the long division algorithm, the remainder r is a whole number with $0 \leq r < b$. If $r = 0$, then the long division algorithm stops at that step, representing a terminating decimal. Otherwise r is a number from the following list: $1, 2, 3, \ldots, b-1$. By the bth step in the long division, some remainder has to appear again. Then, the sequence of digits between these two appearances of the first repeating remainder is the repeating part of the decimal expansion.

Let's follow this through for $6/11$. We start the long division $6 \div 11$: at the first stage we get an approximation of .5 with a remainder of 5; at the next stage we have an approximation of .54 with a remainder of 6. But that is where we started, so we can conclude that continuing this process repeats the consecutive remainders 5, 4 endlessly, and thus the decimal expansion for 6/11 is $0.\overline{54}$.

A *repeating* decimal is a way of representing rational numbers in arithmetic. The decimal representation of a number is said to be repeating if it becomes periodic (repeating its values at regular intervals) and the infinitely-repeated portion is not zero. While there are several notational conventions for representing repeating decimals, none of them are accepted universally. In the United States the convention is generally to indicate a repeating decimal by drawing a horizontal line (a vinculum) above the infinitely-repeated digit sequence. Note, if the infinitely-repeated portion is a zero, the rational number is called a *terminating decimal*, since the zeros can be omitted and the decimal terminates before these zeros.

To illustrate the general pattern, let's consider 1/7. If we calculate $1 \div 7$ we find the remainders are sequentially; $3, 2, 6, 4, 5, 1$. Notice, each is greater than 0 and less than 7. Since there are only six possible remainders, the next step of the long division must produce one of these numbers. In fact, it is the first one 3, so we conclude that $1/7 = 0.\overline{142857}$.

Try this with 26/111. The first step in the long division produces a 0.2 with a remainder of 38, the second step gives 0.23 with a remainder of 44, and the third step a 0.234 with a remainder of 26. Since we started out with $26 \div 111$, this is our first repeater, and so the decimal expansion of 26/111 is $0.\overline{234}$.

Section 1.3: Solve Percent Problems Including Discounts, Interest, Taxes, Tips, and Percent Increase or Decrease.

Solve real-world and mathematical problems involving the four operations with rational numbers. 7.NS.3

Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies. For example: If a woman making $25 an hour gets a 10% raise, she will make an additional 1/10 of her salary an hour, or $2.50, for a new salary of $27.50. If you want to place a towel bar $9\frac{3}{4}$ *inches long in the center of a door that is* $27\frac{1}{2}$ *inches wide, you will need to place the bar about 9 inches from each edge; this estimate can be used as a check on the exact computation. 7.EE.3*

Another kind of fraction is the percentage. The word percent comes from the Latin phrase, per centium, literally "of one hundred" and is a contraction in English of the French pour cent. Long before decimals were used, the need to work with tenths and hundredths was evident. The use of percentages, or similar standards in probability, has been seen in pre-classical China, India, and Egypt; it is believed that the first appearance in India was around 3500 BCE (although this date has been a point of debate amongst historical mathematicians). Although in Rome, taxes were calculated on the basis of three types of fractions. For example, Augustus levied a tax of 1% on goods sold at auction and 4% on every slave sold. While the notation wasn't employed then, computations were often made in fractions which were multiples of 1/100. Over the years computations with a denominator of 100 became standard with interest rate quotes in hundredths. The percent sign was developed with respect to the Italian phrase per cento, short hand for per one hundred.

The use of "percent" is an intuitive way, given our place value system, of speaking about parts of a whole, and are used in grades, sports, surveys andinterest rates, just to name a few. Percents are a special type of fraction with a denominator of 100. For example, if 20% of the 7th grade plans to purchase Val-o-grams on Valentine's Day next year, this tells us that 20 out of every 100 7th graders plan to purchase a Val-o-gram. Hence, 20% is the fraction 1/5, written as a decimal 0.2 or 0.20.

Percents represent fractions or decimals and means, "per hundred." Recall that percentages represent specific parts of a whole, thus 25% means 25 per hundred, $\frac{25}{100}$, or 0.25. The symbol "%" is used to represent percent. So 420% means $\frac{420}{100}$, 4.20, or 420 per hundred.

Why use percents when we could use ordinary fractions? By using the denominator 100, it becomes easy to compare fractional amounts of different quantities. For example, if the fraction of 7th grade students at Pleasant Grove Junior High (PGJH) who like chocolate is $\frac{300}{400}$ and the fraction of 7th grade students at Oak Canyon Junior High (OCJH) who like chocolate is $\frac{315}{450}$, we have to do some calculating to tell which Junior High group of 7th graders has the greater fraction of students who like chocolate. On the other hand, it we are told that 75% of

7th grade students at PGJH like chocolate and 70% of 7th grade students at OCJH like chocolate, then we know immediately that 7th graders at PGJH have a greater fraction of students who like chocolate. So, percents are a way of putting fractions over a common denominator, making comparisons easy.

Since percents are alternative representations of fractions and decimals, it is important to be able to convert among all three forms.

Conversion 1: Percents to Fractions

Using the definition of percent, $25\% = \frac{25}{100}$

Conversion 2: Percents to Decimals

To convert a percent directly to a decimal, "drop the % symbol and move the number's decimal point two places to the left." This is because "percent" means "over 100": 38% means 38/100 which is represented by the decimal .38.

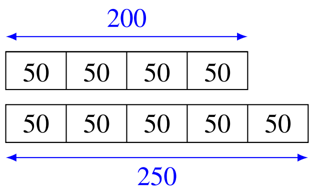
Conversion 3: Decimals to Percents

Here we merely reverse the shortcut demonstrated in case 2 or we rewrite the decimal as an equivalent fraction. For example, $0.25 = \frac{25}{100}$ as a fraction with a denominator of 100, given that the "5" is in the hundredths place. We then utilize the definition of percent and we arrive at 25%. In essence, percents are obtained from the decimals by "moving the decimal point two places to the *right* and writing the % symbol on the right side."

Conversion 4: Fractions to Percents

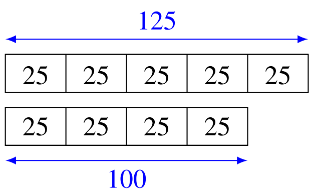
Percent is a fraction with a denominator of 100. Thus, we express $\frac{25}{100} = 25\%$, or $\frac{2}{5} = \frac{2 \cdot 20}{5 \cdot 20} = \frac{40}{100} = 40\%$, and $\frac{3}{25} = \frac{3 \cdot 4}{25 \cdot 4} = \frac{12}{100} = 12\%$. Lastly, fractions can be converted to decimals using long division as we have seen previously.

If a quantity grows, then the increase in the quantity, determined as a percent of the original, is the *percent increase*. If the number of students at Lincoln Middle School that participated in the Science Fair in 2011 was 200, but by 2012 had increased to 250 students, what is the percent increase in the number of students participating in the Science Fair? There were 50 more students who participated in 2012, that is, 50 represents 25% of the original 200 students; therefore, the percent increase was 25% as illustrated in the model.



Additionally, the fraction change was an increase by $\frac{1}{4}$, the percent change was an increase by 25%, the 2012 fractional portion of the original would be represented as they have their 2011 students and lastly, the percent of the original would be represented as 125% of their 2011 students. Furthermore, the fraction change expression, and percent change expression would be illustrated as:

Fraction Change Expression	**Percent Change Expression**
$200(1) + 200\left(\frac{1}{4}\right)$ $= 200\left(\frac{5}{4}\right)$ $= 250$	$200(1) + 200(.25)$ $= 200(1.25)$ $= 250$



If a quantity shrinks, then the decrease in the quantity, determined as a percent of the original, is the percent decrease. If the number of Grade 7 students at Lincoln Middle School that had land-line phones in 2011 was 125, but by 2012 had decreased to 100 land-line phones, what is the percent decrease in the number of land-line phones? There were 25 fewer land-line phones in 2012, that is, 25 represents 20% of the original 125 land-line phones. Therefore, the percent decrease was 20%, as illustrated in the model.

Additionally the fraction change was a decrease by $\frac{1}{5}$, the percent change was a decrease by 20%, the fractional portion of the original would be represented as there are 4/5 the number of land-line phones as in 2011, and lastly, the percent of the original would be represented as 80% of their 2011 students. Furthermore, the fraction change expression, and percent change expression would be illustrated as;

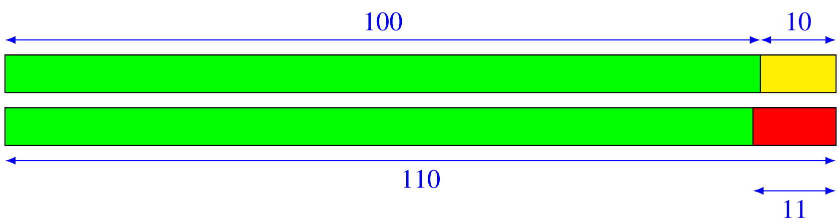
Fraction Change Expression	**Percent Change Expression**
$125(1) + 125\left(\frac{1}{5}\right)$	$125(1) + 200(.20)$
$= 125\left(\frac{4}{5}\right)$	$= 125(.80)$
$= 100$	$= 100$

Can percentage increase can be "reversed" by the same percentage decrease?

Example 8.

Start with 100 and do a 10% increase. Follow this by a 10% decrease. Are you back at 100? Explain.

Solution. 10% increase from the original 100 is an increase of 10, which equals 110. But a 10% reduction from 110 is a reduction of 11 (10% of 110 is 11), which means we ended up at 99 (not the 100 we started with). What happened? The 10% took us up 10, then 10% took us down 11, that is, the 10% increase was applied to 100, but the 10% decrease was applied to 110 as illustrated below.



Chapter 2. The Number System (Integers and Rational Numbers)

In this second chapter, students extend and formalize their understanding of the number system, including negative rational numbers. Students first develop and explain arithmetic operations with integers, using both tiles and the number line as models. Then, they move, in both models, to the rational numbers by observing that the unit represented by the tile, or the first hash mark on the number line, can be changed; for example it could be the qth part of the unit originally represented by the first tile, or the first hash mark on the line. As the laws of arithmetic remain the same, they extend directly to the arithmetic for the qth part of the original unit. It all holds together from the simple observation that q copies of the qth part of the unit returns us to the original unit. In this way, the arithmetic of rational numbers is a consequence of the properties of the four basic operations of addition, subtraction, multiplication and division.

By applying these properties, and by viewing negative numbers in terms of everyday contexts (i.e. money in an account or yards gained or lost on a football field), students explain and interpret the rules for adding, subtracting, multiplying, and dividing with negative numbers. Students re-examine equivalent forms of expressing rational numbers (fractions of integers, complex fractions, and decimals) and interpret decimal expansions in terms of successive estimates of the number representing a point on the number line. They also increase their proficiency with mental arithmetic by articulating strategies based on properties of operations.

In Grade 6 students understand that positive and negative numbers are used together to describe quantities having "'opposite" directions or values, learning how to place and compare integers on the number line. They use positive and negative numbers to represent quantities in real-world contexts, explaining the meaning of 0 in each situation. Students recognize signs of numbers as indicating locations on opposite sides of 0 on the number line and recognize that the opposite of the opposite of a number is the number itself.

The development of rational numbers in Grade 7 is the beginning of a development of the real number system that continues through Grade 8. In high school students will extend their understanding of number into the complex number system. Note that in Grade 7 students associate points on the number line to rational numbers, while in Grade 8, the process reverses: they start with lengths and try to identify the number that corresponds to the length. For example, students find that the length of the hypotenuse of a right triangle is $\sqrt{5}$, they then try to identify the number that corresponds to that length.

The main work of this chapter is in understanding how the arithmetic operations extend from fractions to all rational numbers. At first addition is defined for the integers by adjunction of lengths, and subtraction as the addition of the opposite; in symbols: $a - b = a + (-b)$. These idea is a prelude to vector concepts to be developed in high school mathematics. Students' understanding of multiplication and division is extended to integers so that their properties continue to hold (using the understanding that the a number is the opposite of its opposite(e.g., $-(-1) = 1$).

Section 2.1: Add and Subtract Integers; Number Line and Chip/Tile Models

Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram. 7NS.1.

Describe situations in which opposite quantities combine to make 0. For example, a hydrogen atom has 0 charge because its two constituents are oppositely charged. 7.NS.1a

Understand $p + q$ as the number located a distance $|q|$ from p, in the positive or negative direction depending on whether q is positive or negative. Show that a number and its opposite have a sum of 0 (are additive inverses). Interpret sums of rational numbers by describing real-world contexts.7.NS.1b

Understand subtraction of rational numbers as adding the additive inverse, $p - q = p + (-q)$. Show that the distance between two rational numbers on the number line is the absolute value of their difference, and apply this principle in real-world contexts.7NS.1c.

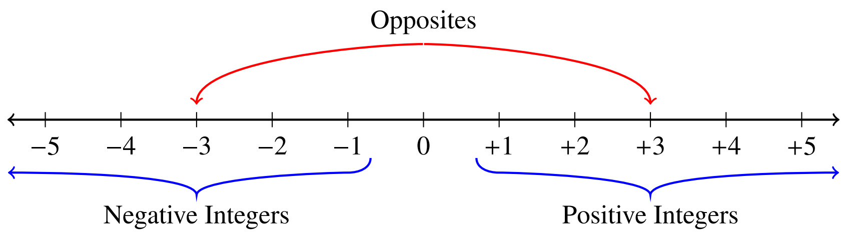
We start the study of the operations of addition and multiplication for integers with a review of 6th grade, where the integers are place on the number line. The reason for this is that, once students have understood the arithmetic operations with integers, in particular, the rules of signs, then the extension to all rational numbers is natural. This is because the rational numbers are placed on the just by a change of unit from the unit interval to the $1/q$th interval for every positive integer q (see section 3). The only complication is the addition of rational numbers; this is discussed at the end of section 3.

The goal in this section is to build intuition and comfort with integer addition and subtraction so that by the end of the section students can reason through addition and subtraction of integers without a model, and then will be able to extend those operations to all natural numbers.

Students start by working with "opposites" (additive inverses) to notice that pairs of positives and negatives result in "zero pairs." They then move to adding integers. Students know from previous grades that the fundamental idea of addition is "joining." After reviewing prior understanding of subtraction, we turn the number line model of subtraction (as adding the opposite). Students should notice that joining positive and negative numbers often results in zero pairs and that which is "left over" is the final sum. Students will develop this idea first with a chip or tile model and the real line.

Rules for operating with integers come from the permanence of the rules of arithmetic, by which we mean that extensions of the concept of number require that the rules of arithmetic continue to hold. Students should understand that arithmetic with negative numbers is consistent with arithmetic with positive numbers.

When looking at a number line, two numbers are opposites when they are the same distance away from zero, but in opposite directions. Those numbers to the right of 0 are the positive numbers, and those on the left are the negative numbers. For example, "3" represents the point that is 3 units to the right of 0, and "−3" is its opposite, three units to the left. Two integers are opposites if they are each the same distance away from zero, but on opposite sides of the number line. One will have a positive sign, the other a negative sign. In the number line, +3 and −3 are labeled as opposites (it is customary to not write the plus sign in front of a positive number, so +3 will be denoted by 3.



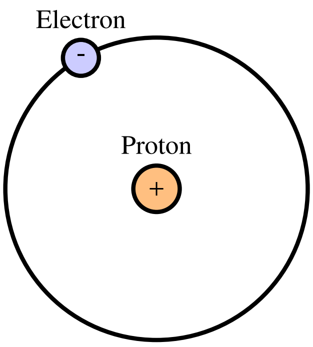
The full number line appears naturally in measurements, such as the thermometer, elevation, timelines, and banking. In each of these applications, we are making measurements "less than zero" of which the number line illustrates each of these concepts. Three observations that are important to notice:

The integers are ordered: we say *a is less than b*, and write $a < b$, whenever b is to the right of a on the number line.

The absolute value $a|$ of an integer a is the distance from the point on the line to 0. A number and its opposite have the same absolute value.

Every natural number has an opposite, or additive inverse. The negative integers are opposites of the positive integers.

For example, the opposite of 5 is -5, and the absolute value of both numbers is 5. Since the positive integers are the opposites of the negative integers, we conclude that $-(-5) = 5$, and in general $-(-a) = a$ for any integer a.The opposite of 0 is 0.



Hydrogen Atom. Source: www.jwst.nasa.gov/firstlight.html

The idea of "opposite" occurs in the real world in many ways. The opposite of "income" is "debt." If I find \$5, I can claim that I have +5 dollars; if I lose \$5, that amounts to having -5 dollars. In chemistry a hydrogen atom, like other atoms, has a nucleus. The nucleus of a hydrogen atom is made of just one proton (a positive charge). Around the nucleus, there is just one electron (a negative charge), which goes around and around the nucleus. As a result, the "positive" proton and the "negative" electron balance the electrical charge, so that the hydrogen atom is electrically neutral (basically zero).

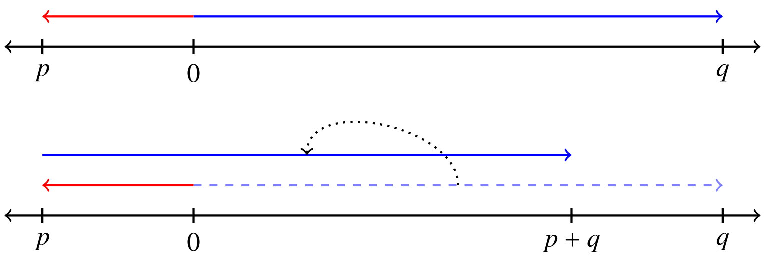
As with any new topic, it is important to start with familiar contexts so that students can use prior knowledge to build meaning. With integers, students often get confused about which symbol is the operation or in which direction they are moving when they compute, so having a context is particularly important. As students learn to compare and compute, they can use the contexts to ground their thinking and justify their answers.

In this course we will understand integers as points on the number line, and will also model the operations on integers with the chip/tile model. We emphasize the number line model because it is key to understanding the extension of arithmetic, and it provides the bridge between algebra and geometry that has been central to the development of mathematics for a millennium. The number line model amounts to working with a definition of addition and subtraction for integers and subsequently, rational numbers; and the chip/tile model (quantity) amounts to working with a theorem about addition and subtraction ($a - a = a + (-a) = 0$, represented by "zero pairs"), as a basis for understanding how the properties of operations should extend to integers.

A number line model has several advantages. In Grade 6 students learned to locate and add whole numbers, fractions and decimals on the number line. Additionally, the number line is an important connection to the coordinate axis, which involves two perpendicular numbers lines, explored in Grade 8.

Previously addition was represented by linking the line segments together. With integer addition, the line segments have directions, and therefore a beginning and an end. Thus, integers involve two concepts: length of a line segment and its direction.

In the number line model arrows are used to show distance and direction. The arrows help students think of integer quantities as directed distances. An arrow pointing right is positive, and a negative arrow points left. Each arrow is a quantity with both length (magnitude) and direction (sign). To put the integers on the line, we first select a point to the right of 0 and designate it as 1. The point at the same distance from 0, but to the left, is -1, the opposite of 1. Now, for a positive integer p, move to the right a distance of p units: the end point represents p. For a negative integer, do the same thing, but on the left side of 0. Now, we add integers on the line by adjunction of the corresponding directed line segments. To add the integers p and q: begin at zero and draw the line segment (arrow) to p. Starting at the endpoint p, draw the line segment representing q. Where it ends is the sum $p + q$. If p and q are of the same sign, the point $p + q$ is farther from the origin than q. If the second line segment is going in the opposite direction to the first, it can backtrack over the first, effectively cancelling part or all of it out. The following figure depicts the sum $p + q$ where p is negative and q is positive.



Example 1.

Demonstrate each of the following on a number line.

a. $2 + 3$

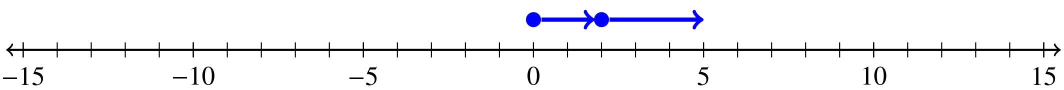
b. $-4 + 3$

c. $12 + (-9)$

d. $-2 + (-3)$

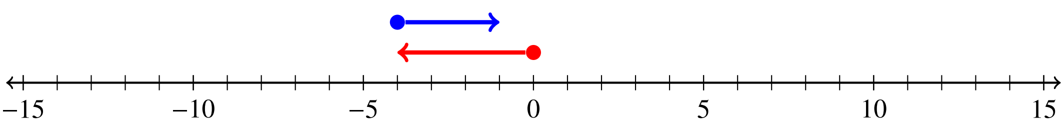
Solution. To avoid confusion, students should start by circling the operation, which will be relevant in preparation for subtraction. It is important to emphasize the difference between an operation (add, subtract, multiply, or divide) and a positive or negative number. This will be an issue later, so it is vital to start addressing this as early on.

a. $2 + 3$: this addition can be thought of as starting at 0 and counting 2 units to the right (in the positive direction on the number line) and then "counting on" 3 more units to the right.



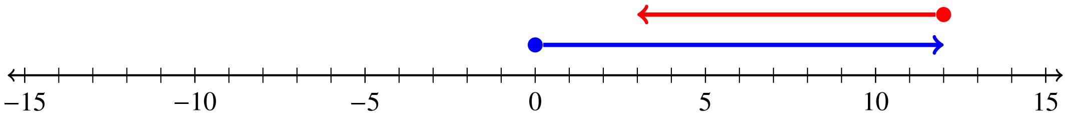
$2 + 3 = 5$

b. $-4 + 3$: this addition can be thought of as starting at 0 and counting 4 units to the left (in the negative direction on the number line) and then "counting on" 3 more units to the right.



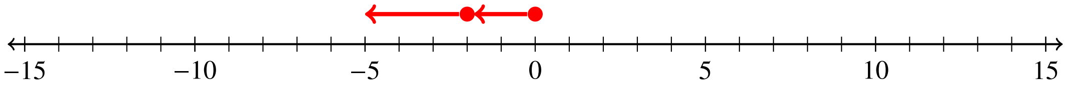
$-4 + 3 = -1$

c. $12 + (-9)$: start at 0 and count 12 units to the right, then count on 9 more units to the left.



$$12 + (-9) = 3$$

d. $-2 + (-3)$: Here we count 2 units to the left, and then 3 more units to the left.



$$2 + 3 = 5$$

Finally, note that if we add the opposite of a number to a number, we return to 0:

$$n + (-n) = 0 = (-n) + n\,.$$

for any number n. This equation states that n and $-n$ are *additive inverses,* that is, $-n$ and n add up to 0. Finally, note that parentheses can be used to avoid strange, or ambiguous expressions like $- - 3 = 3$ and $3 + -3 = 0$. Instead, write $-(-3) = 3$ and $3 + (-3) = 0$. Even 0 has an opposite, namely, itself, since $0 + 0 = 0$; that is, $-0 = 0$.

The second model mentioned utilizes colored chips/tile to represent integers, with one color representing positive integers and another color representing negative integers; subject to the rule that chips of different colors cancel each other out. Historically, Chinese traders used a system of colored rods to keep records of their transactions. A red colored rod denoted credits and a black colored rod denoted debits (opposite of how we view debits/credits today). Assign a red chip to represent -1 (debit or a negative charge) and a yellow chip to represent 1 (credit or a positive charge):

 = -1, = 1

A pair of red and yellow chips is called a *zero pair*, because it represents the integer 0. We use the informal term "zero pair" to refer to two things that are "opposite" or "undo each other". The sum of the two elements of a zero pair is zero. We see zero pairs in many contexts. For example, an atom with 5 protons and 5 electrons has a neutral charge, or a net charge of zero.

Example 2.

I took 4 steps forward and 3 steps back. What is the result in words, and how many zero pairs are there?

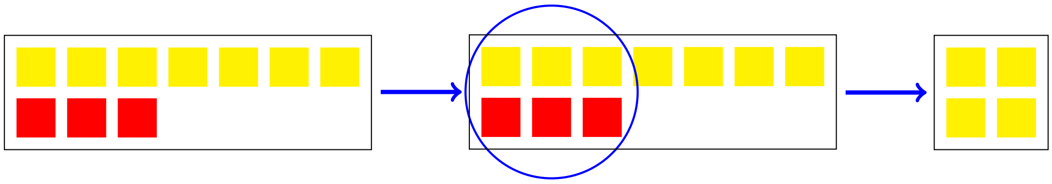
Solution. A visual model is shown and depicts that I am one step ahead from where I stared and that there are 3 zero pairs.

We emphasize that = 0, expressing $(-a) + a = 0$, the additive inverse property of integer addition. A red tile coupled with a yellow tile is a zero pair, that is, they cancel each other.

Example 3.

Demonstrate the addition $-3 + 7$ with the chip/tile model.

Solution. We begin with 7 yellow chips and 3 red chips in our first box. In the second box we recognize that there are three sets of zero pairs circled with each zero pair representing 0; resulting in (in the last box) four yellow chips or 4.



It is important to understand that it is always possible to add to or remove from a collection of any number of pairs consisting of one positive and one negative chip/tile without changing the value (i.e. it is like adding equal quantities of debits and credits). We also recognize that,

$$-3 + 7 = -3 + (3 + 4) = (-3 + 3) + 4 = 0 + 4 = 4 \,.$$

Although not explicitly stated, we are moving students towards understanding, either with the chip/tile model or the number line model the properties of arithmetic. Implicit in the use of chips is that the commutative and associate properties extend to integers, since combining chips can be done in any order.

Integer chips are useful in understanding operations with rational numbers, up to a point. By changing the unit concept from the interval $[0, 1]$ to the $1/q$th part of that interval (for any integer q) we can illustrate, with chips, some of the arithmetic facts for rational numbers. However, if we work with rational numbers with different denominators, this model becomes unwieldy. For this reason, in section 3, where we turn to arithmetic with rational numbers, we will work exclusively with the line model.

Next, students move to subtraction, first by reviewing, from previous grades, that there are two ways to think concretely about subtraction. a) "Take-away:" Anna Maria has 5 gummy bears. José eats 3. How many gummy bears does she now have? $5 - 3$, thus 2 gummy bears. b) "Comparison:" Anna Maria has 5 gummy bears, José has 3 gummy bears, how many more gummy bears does Anna Maria have than José? Again, the operation is $5 - 3$ resulting in 2 gummy bears. We build on the concept of comparison as a concrete way to think about subtraction with integers on the real line. We start by locating integers on the line and note that when comparing two integers, there is a directional or signed distance between them e.g. when comparing 5 and 3 we can think "5 is two units to the right of 3,"($5-3$ is 2), or "3 is two units to the left of 5," ($3-5$ is -2). In 2.1e students will examine subtraction exercises and notice that $a - b$ can be written as $a + (-b)$ and that $a - (-b)$ can be written as $a + b$:

$$a - b = a + (-b) \qquad a - (-b) = a + b \,.$$

As an example: if we subtract 11 from 7, we move 11 units to the left from 7, to -4. That is the same as adding -11 to 7.

Example 4.

Model $7 - 2$ and $2 - 7$ on the number line.

Solution. Think in terms of addition: $7 - 2 = 7 + (-2)$, and $2 - 7 = 2 + (-7)$.

a. Locate 7 and -2 on the number line. Take the directed line segment corresponding to -2 and move it so that its beginning point is at the point 7. Then the endpoint lands at 5.

b. Locate 2 and -7 on the number line. Take the directed line segment corresponding to -7 and move it so that its beginning point is at the point 2. Then the endpoint lands at -5.

Describing subtraction in terms of the number line model brings home the realization that integers involve two concepts: magnitude and direction, sometimes designated as "directional distance." We now see that there is a distinction between the distance from a to b and how you get from a to b, in other words, the directional distance.

Adding and subtracting integers can cause students a great deal of trouble particularly when they first confront exercises that have both addition and subtraction with integers. For example, in expressions like $-5 - 7$ students are often unsure if they should treat the "7" as negative or positive integer and if they should add or subtract. Put another way, students are not sure that $-5-7 = -5+(-7)$, and worry that it could mean $-(5-7)$, or even $-5-(-7)$. For this reason, it is extremely helpful that, from the beginning of this chapter, students articulate what they think the expression means.

We close section 2.1 with a final word of caution. Subtraction in the set of integers is neither commutative nor associative: $5 - 3 \neq 3 - 5$ because $3 \neq -3$; $(5 - 2) - 1 \neq 5 - (2 - 1)$ because $2 \neq 4$.

The expression $5 - 2 - 1$ is ambiguous unless we know in which order to perform the subtractions. The convention is that $5 - 2 - 1$ means $(5 - 2) - 1$; that is; left to right, following from order of operations in Grade 5. However, it is best to get in the habit of using parentheses to eliminate ambiguities.

Section 2.2: Multiply and Divide Integers; Represented with Number Line Model

Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers. 7.NS.2.

Here we restrict attention to integers; in the next section we shall move to rational numbers, and deal with the full standard in that context. . The goal here for students in this section is twofold: 1) fluency with multiplication and division of integers and 2) understanding how multiplication and division with integers is an extension of the rules of arithmetic as learned in previous grades.

Multiplication of integers is an extension of multiplication of whole numbers, fractions and decimals.

Let us consider a typical elementary multiplication problem.

Example 5.

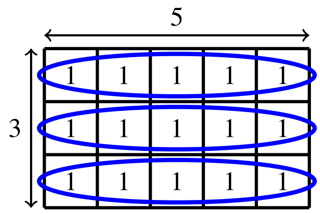
Sally babysat for her mother for 3 hours. Her mother paid her \$5 each hour. How much money did Sally earn after 3 hours?

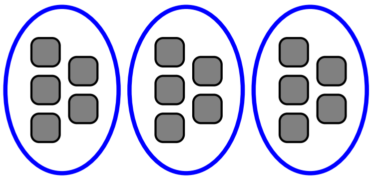
Solution. We note that Sally earned \$5 + \$5 + \$5 = \$15, illustrated as

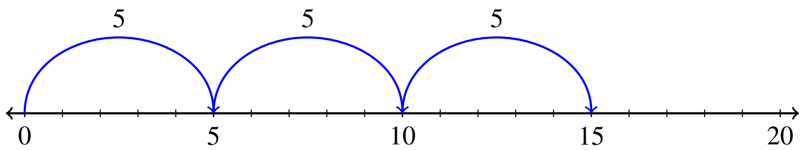
Source: http://www.newmoney.gov/currency/5.htm

which is the sum of 3 fives, written as $3 \cdot 5$ and is read as "3 times 5," and means the total number of objects in 3 groups if there are \$5 in each group.

Additional representations include:







The basic definition of multiplication of whole numbers is, "if a and b are non-negative numbers, then $a \cdot b$ is read as "a times b," and means the total number of objects in a groups if there are b objects in each group." Note: if we rotate the rectangle in the first representation, we will have 5 groups of 3 objects in each group, representing 5×3. This is a way of seeing that $a \cdot b = b \cdot a$ for any positive a and b.

In elementary grade the dot symbol for multiplication is replaced by a cross, $\times$ (not to be mistaken for the letter x or x), or by an asterisk * (used in computer programming). Sometimes no symbol is used (as in ab), or parentheses are placed to distinguish the factors such as $(a)(b)$. In this text we will typically use the dot or parenthesis to express multiplication of expressions and the cross for numbers, to be sure that 3×5 is distinguished from 35.

We note that a and b are called factors, and that the first factor a stands for the numbers of groups, and the second factor b stands for the numbers of objects in each group (although, as we have pointed out, those roles are interchangeable).

How can you tell if the operation needed to solve a problem is multiplication? To answer this question we consider the definition of multiplication once again. Multiplication applies to situations that involve equal groups. In the illustration for our previous problem, there were 3 distinct groups of \$5 in each group. Thus, according to the definition of multiplication, there are 3 ? 5 dollar bills in all. Simply put, whenever a collection of objects is arranged into a groups, and there are b objects in each group, then we know that according to the definition of multiplication, there are $a \cdot b$ total objects. Up to this point we have examined multiplication only for whole numbers, so, what is the meaning of multiplication of negative numbers?

Understand that multiplication is extended from fractions to rational numbers by requiring that operations continue to satisfy the properties of operations, particularly the distributive property, leading to products such as (-1)(-1) = 1 and the rules for multiplying signed numbers. Interpret products of rational numbers by describing real-world concepts. 7.NA.2a

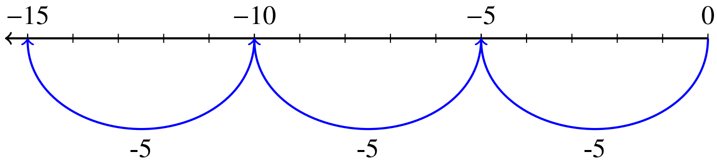
We will discuss this standard here in the context of integers, and then in the next section extend that logic to all rational numbers.

Example 6.

What does $3 \times (-5)$ mean?

Solution. Just as 3×5 can be understood as $(5) + (5) + (5) = 15$, so $3 \times (5)$ can be understood as $(-5) + (-5) + (-5) = -15$.

On the number line, we think of 3×5 as 3 jumps to the right (or up) on the number line, starting at 0. Similarly, $3 \times (-5)$ is represented by 3 jumps of distance 3 *to the left* starting at 0, as in the following figure:

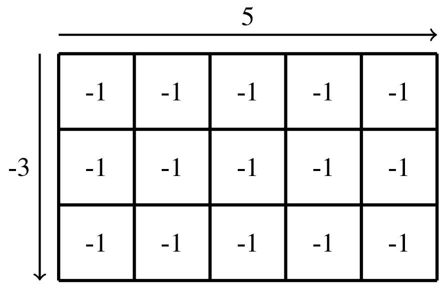


Using chip/tiles, $3 \times (-5)$ means 3 groups of 5 yellow tiles. And finally in context: if I lose \$5 on three consecutive days, then I've lost \$15.

Example 7.

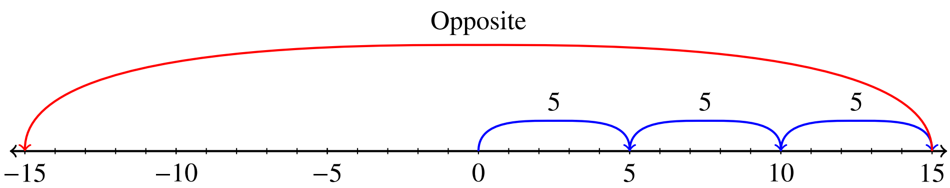
What does $(-3) \times 5$ mean?

Solution. There are two ways a student may attack this using rules of arithmetic: one way is to recognize that $(-3) \times 5$ is the same as $5 \times (-3)$ (e.g. the commutative property) and then apply the logic above: $-3 + -3 + -3 + -3 + -3 = -15$. A model would be illustrated as:



Another is to turn to the understanding of integers from 6th grade. There the idea of -3 was developed as the "opposite" of 3. Now , if the associate law of arithmetic extends, (the opposite of 3)$\times$5 is the same as (the opposite of 3×5), or -15.

On the number line, we record $(-3) \times 5$ as the opposite of 3×5:



Example 8.

Let us note that this argument gives us the equation $(-1) \times (-1) = 1$. For, multiplication by -1 is reflection to the opposite side of 0, and therefore lands on the opposite point. So, this equation is simply saying that he opposite of -1 is 1. It is instructive to go through the reasoning for this multiplication table:

$\times$	**1**	**−1**
1	1	−1
−1	−1	1

In other words, the product of two numbers with the same sign is positive, and if the numbers have opposite signs, it s negative. Notice that this is the same rule as that for the sum of even and odd numbers: the sum of two numbers of the same parity (both even or both odd) is even, if the parity is different, the sum is odd.

Understand that integers can be divided, provided that the divisor is not zero, and every quotient of integers (with non-zero divisor) is a rational number. 7.NS.A.2b

Just as the relationship between addition and subtraction helps students understand subtraction of rational numbers, so the relationship between multiplication and division helps them understand division. To make this more precise: for addition, the "opposite" of a number a is $-a$, the solution of the problem $a + x = 0$. On the number line, a represents going out from 0 to the point a, and adding $-a$ brings us back to the beginning. For this reason $-a$ is called the "additive inverse" of a. So, for multiplication, the "opposite" of a is the solution of the equation $ax = 1$, denoted $1/a$. On the number line, multiplying 1 by a means adding 1 to itself b times. To invert this process (that is, to get back to 1), we have to partition the result into a pieces of the same length; that is the partitive definition of division. So, dividing by a undoes multiplying by a, so $1/a$ is the "multiplicative inverse" of a; that is $a \times (1/a) = 1$, or $1/a$ is the solution to the equation $ax = 1$. It is this logic that extends from fractions to all rational numbers, as we shall see in the next section.

One important point: since $0 \times x = 0$ for every number x, there is no solution to the equation $0 \times x = 1$; and it is for this reason that *we cannot divide by zero*: it just plain does not make any sense.

In short, in terms of the number line, the properties of the operations of multiplication and division carry over from fractions to all numbers, positive and negative. Let us illustrate.

Example 9.

a. Calculate $(-15 \div 5$.

b. Calculate $8 \div (-4)$.

c. Calculate $(-12) \div (-3)$.

Solution.

a. Since division by 5 is the inverse of multiplication by 5, the equation tells us that the solution is some number x, which when multiplied by 5 gives us -15. But we know that $(-3) \times 5 = -15$, so the answer has to be -3.

b. If $a = 8/(-4)$, then a is the solution of the problem $-4x = 8$. So, a has magnitude 2, but is it negative or positive? Let's check: $(-4)(2) = -8$ and $(-4)(-2) = 8$. So $a = -2$.

c. $(-12) \div (-3)$ is that number which, when multiplied by -3 gives us -12. So, it has to be negative (for the product of two negatives is positive, so it can't be positive) and it has to be of magnitude 4. Therefor the answer is -4.

Section 2.3: Add, Subtract, Multiply, Divide Positive and Negative Rational Numbers (all forms).

Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram. 7.NS.1.

Understand $p + q$ as the number located a distance $|q|$ from p, in the positive or negative direction depending on whether q is positive or negative. Show that a number and its opposite have a sum of 0 (are additive inverses). Interpret sums of rational numbers by describing real-world contexts. 7.NS.1.b.

Understand subtraction of rational numbers as adding the additive inverse, $p - q = p + (-q)$. Show that the distance between two rational numbers on the number line is the absolute value of their difference, and apply this principle in real-world contexts. 7NS.1.c.

Apply properties of operations as strategies to add and subtract rational numbers. 7NS.1.d.

Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers.7NS.2.

Understand that multiplication is extended from fractions to rational numbers by requiring that operations continue to satisfy the properties of operations. 7NS.2a.

Understand that integers can be divided, provided that the divisor is not zero, and every quotient of integers (with non-zero divisor) is a rational number. If p and q are integers, then $-(p/q) = (-p)/q = p/(_{q})$. *Interpret quotients of rational numbers by describing real world contexts. 7NS.2b.*

Apply properties of operations as strategies to multiply and divide rational numbers. 7.NS.2c.

Now we move on to rational numbers: placing them as points on the number line, and then extending the understanding of arithmetic operations to all natural numbers.

In the sixth grades students learned how to place the rational numbers on the number line, and we shall depend upon that knowledge in this section, although we will start anew: by defining and working with addition and subtraction of points on the line with a specific point identified as 0, by considering the point, and the line segment from 0 to that point as one and the same thing.

The points on one side of 0 are considered "negative" and those on the other side as "positive." Ordinarily the line is drawn as a horizontal line, and the positive points are those on the right of 0. If the line is drawn vertically, then the custom is to designate the segment above 0 as the positive segment. In some contexts the number line may be neither horizontal or vertical; it is important to realize that its position in the plane (or, for that matter, in space) is not relevant; all that matters is that a base point has been designated as 0, and one side has been designated as "positive." Once that has been done, the concept of "opposite" (the opposite of a is the symmetric point on the other side of 0). We use the notation $-a$ to represent the opposite of a. Without actually using the term *vector*. it helps to use this idea to understand *opposite*, and the arithmetic operations of addition and subtraction. For example, the act of "taking the opposite of a" is performed by reflecting in the origin, as illustrated in figure 1.

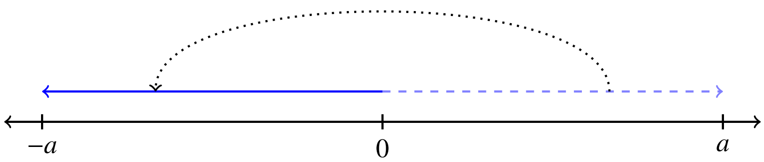


Figure 1. Taking the opposite: $a \to -a$

Addition, denoted $a + b$ for two points a and b on the line, is defined (vectorially) by adjunction: the sum of a and b is the endpoint of the interval formed by adjoining the interval to b to the interval to a (see figure 2).

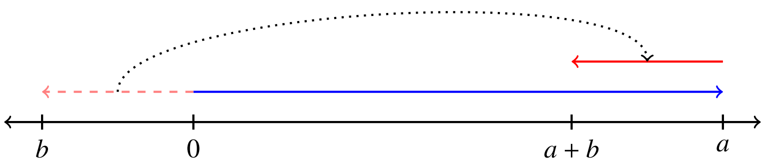


Figure 2. Addition: $(a, b) \to a + b$

This brings the integers into the picture in this way: for any integer n, and any point a on the line, na is the endpoint obtained by appending the interval $[0, a]$ to itself n times. For $n > 0$, this is clear, and for $n < 0$ we mean the adjunction on the opposite side of $[0, a]$. In other words, for $a > 0$ and $n > 0$, na is the endpoint of the interval starting from 0 and moving to the right a distance of n lengths of size a, and $(-n)a$ is the endpoint of the interval starting from 0 and moving to the left a distance of n lengths of size a. This can be summarized by the equation (which works for either n or a either positive or negative:

$$(-n) \times a = n \times (-a) = -(n \times a)$$

Subtraction, denoted $a - b$, for two points a and b on the line, is defined as $a - b = a + (-b)$. Often, students will be confused by the symbol "$-$", since it represents both "subtraction" and "the opposite ." A to deal with this confusion is to understand that it makes sense if we think of the opposite of a as obtained by subtracting a from 0: $-a = 0 - a$. In fact, in doing addition and subtraction, it is customary to delete the addend 0 if it appears. For example, one might replace $5 - 5 + a$ by $0 - a$ and then just omit the 0, getting $-a$. The basic fact that resolves the confusion is this: if b is subtracted from a, that is the same as adding $-b$ to a; that is, $a - b = a + (-b)$. Furthermore, $-(-a) = a$, and $a - (-b) = a + b$.

Having defined addition and subtraction for points (as intervals) on the real line, we can pictorially demonstrate that the properties of addition extend naturally to the same operations on the line. Here are two examples; students can reinforce their understanding of those operations by drawing diagrams for other laws of arithmetic.

Example 10.

$\mathbf{a + b = b + a}$

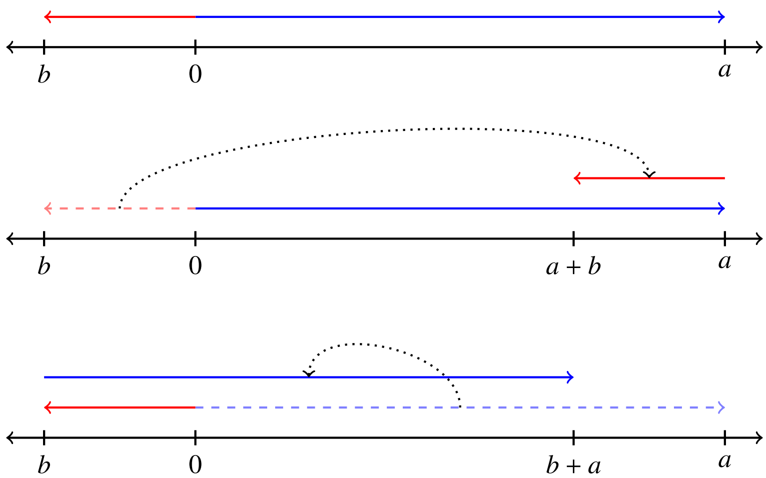
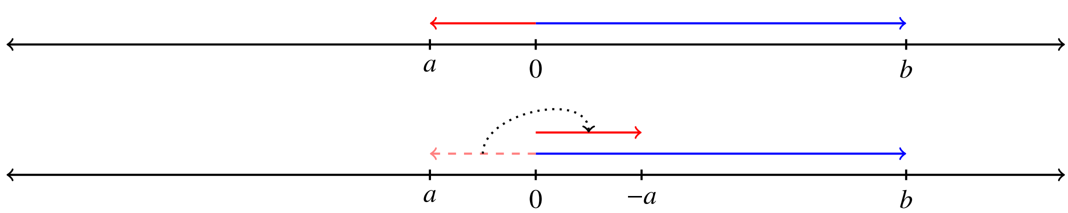


Figure 3. $a + b = b + a$

Example 11.

$\mathbf{-(b - a) = -b + a}$



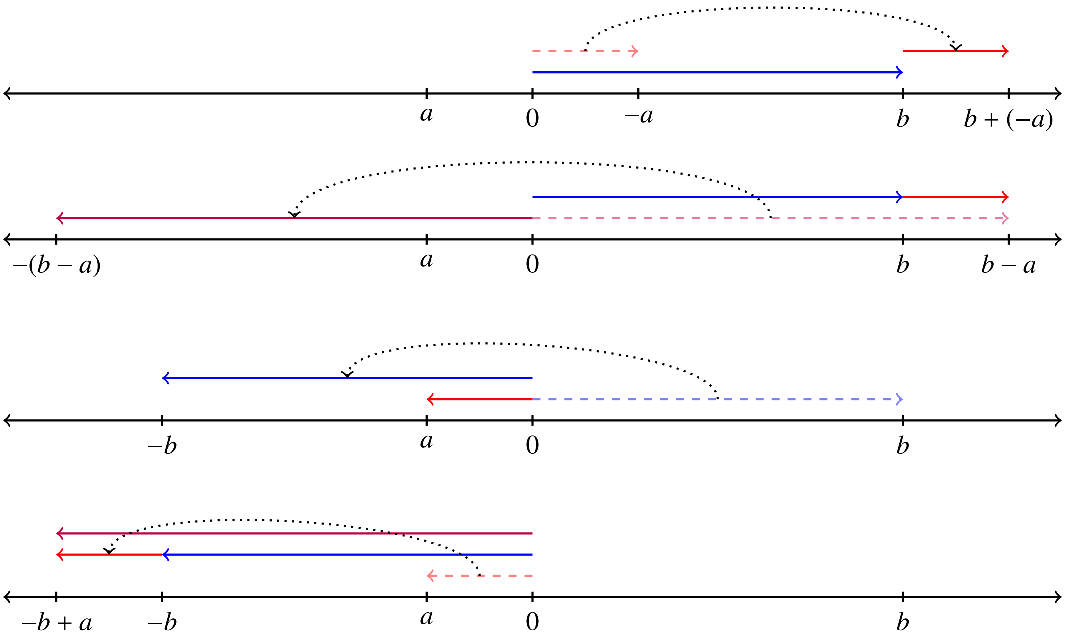


Figure 4. $-(b-a) = -b + a$

Bringing this discussion together with that of the previous section, we can summarize in this way:

Take a line, and a point 0 on the line. Let's assume the line is horizontal. The point 0 divides the line into two segments; the "positive side" on the right of the point 0, and the "negative side" on the left. Define *opposite* and *addition* as above. There is a relation on this line: the expression $a < b$ means that the point a lies to the left of b. This is the same as saying $b - a$ is positive. For any two different points a, b, either $a < b$ or $b < a$. If $a < b$ and $b < c$, then $a < c$. For any a and b, $a \leq b$ the segment, or interval, $[a, b]$ is the set of points between a and b, ordered from a to b. In particular, $[a, a]$ is just the point a.Finally we define addition of points a and b by adjunction of the intervals $[0, a]$ and $[0, b]$: put the 0 endpoint of $[0, b]$ at the 0 endpoint of $[0, a]$: the combined interval is the interval $[0, a + b]$.

Now select a point on the right of 0, and denote it as 1. Then, for any positive integer n, n is represented by the sum of n copies of the interval between 0 and 1 (where, for n negative, we do the same on the other side of the line starting with the opposite -1 of 1) .

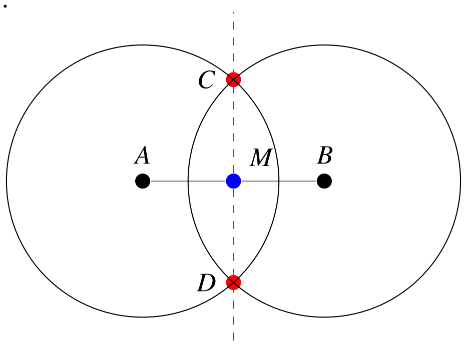


Figure 5

Finally, we put rational numbers on the line. Given p/q, where p is an integer and q is a positive integer, first partition the interval $[0, 1]$ into q equal parts, and denote the first interval as $[0, 1/q]$. If we adjoin q copies of this interval to each other, we end up at the point 1, showing that $q(1/q) = 1$. Now p/q is the endpoint of the interval obtained by adjoining p copies of $[0, 1/q]$ to each other (on the left side of 0 if $p < 0$. Incidentally, the instruction "partition the interval $[0, 1]$ into q equal parts" can actually be done by construction with straight edge and compass. For $q = 2$, students probably already know how to bisect a line segment (see figure 5 for a reminder); a different construction for any positive integer q will be learned in high school mathematics.

Multiplication and division of rational numbers, as points on the line, are defined geometrically just as they are on the integers, with the requirement that the rules of arithmetic persist. Let us explain:

First of all, if p is a positive integer and a is a point on the line, Then $p \times a$ is defined as the adjunction of p copies of the interval $[0, a]$ from 0 to a. We can extend the definition for $p = 0$: $0 \times a$ represents zero copies of a; in other words $0 \times a = 0$. Now, if p is a negatiive integer, $p \times a$ is defined as the adjunction of p copies of the opposite of a. Now, if q is a positive integer a/q is defined as the first piece of a partition of the interval $[0, a]$ into q pieces of the same length.

This suffices to define multiplication of any length by an integer, and division of any lengths by a positive integer. But we want to show that we can multiply any two rational numbers, and divide any rational number by a nonzero rational number. To get there takes a few steps.

We are used to the understanding (for a positive integer p, and an interval $[0, a]$, that $p \times a$ represents p copies of $[0, a]$ adjoined together sequential, and $(1/q) \times a$ represents one qth part of $[0.a]$. But that is how we defined a/q, so

$$\frac{1}{q} \times a = \frac{a}{q} \ .$$

Now, how do we multiply any point a on the line by a rational number p/q? If the laws of arithmetic are to hold, we must have

$$\frac{p}{q} \times a = p \times (\frac{1}{q} \times a) = p \times \frac{a}{q} \ .$$

When we defined division, $a \div b$, we said that the divisor b should be not zero. Why not?

We know that division is the inverse operation of multiplication. For example, the unique (one and only one) solution of 15/5 = 3, since 3 is the value for which the unknown quantity in (?)(5) = 15 is true. But the expression 15/0 requires a value to be found for (?) that solves the equation: (?)(0) = 15. But any number multiplied by 0 is 0 and so there is no number that solves the equation. Maybe you would like to introduce a new number, call it ξ, that solves this equation. But this would be a ξ specific to 15, so for every number we need its ξ. This approach is doomed to failure: there is no way to try to define division by 0 so that the laws of arithmetic continue to hold. Furthermore, we want all numbers to have a geometric representation: where on the number line are we going to put all these ξs?

The point here is this: if we want to extend the number field, it should be based on a specific context; in our case: the number line. Let's look at the number line context for our ξs. Locate $15/a$ with $a > 0$ on the number line. As a gets smaller and smaller, $15/a$ gets larger and larger, and if we slide a to zero, $15/a$ slides off the number line on the right. So, $15/0$, if it can make any sense at all, is off the number line. Until we find a new context where this makes sense, we have to abandon the hope to define division by zero.

But, now, what sense are we to make of division by fractions? What is the answer to the question

$$\frac{\frac{1}{5}}{\frac{1}{7}} = ?$$

Let the laws of arithmetic be our guide: the question mark s filled with a number that his this property: when multiplied by $1/7$, we get $1/5$. That is the same as saying: the question mark s filled with a number that his this property: when divided 7, that is, when partitioned into 7 equal parts, each part is a copy of 1/5. The answer then is 7 copies of 1/5, or 7/5:

$$\frac{\frac{1}{5}}{\frac{1}{7}} = \frac{7}{5}\ .$$

In conclusion, these constructions associate to every rational number (quotient of an integer by a positive integer) a point on the line. The question might arise: does this give us *every* point on the plane? In practice, a point is a blot on the line, and so has positive length. So, we can get within that length with the decimal approximation of the point (that is, getting within the nearest integer of the point, then the nearest tenth, then the nearest hundredth etc.). If all we care about is naked-eye accuracy, we might need two or three decimal points. For accuracy with a magnifying glass, we;ll need more; with a microscope, much more, and so forth. No matter how good our instrumentation, a point will appear as a blot. Plato, who invented the concept of the ideal, conceived of this process going on indefinitely - for any finite number of steps, there will always be a blot. But in the ideal, a point has no dimension.

Another way to express this question is through decimal expansions. Given any point, our construction of the decimal approximation to the point can get us as close as we need to the point (as 0.333333333 is close to 1/3, and if we append more 3;'s we continually get closer - but never there). To paraphrase Plato's conception (a paraphrase since decimal expansions were not known at Plato's time) - on the contrary: $0.\overline{3}$ *is* there. Well that's ok, since 1/3 is a rational number.

But what about

$$0.101001000100001000001\cdots ?$$

Here "..." means that the pattern continues indefinitely; the patter being that each 1 is followed by one more 0 than the preceding 1, going on indefinitely. Now a rational number (as we have seen in chapter 1) has a decimal expansion that is either terminating or repeating. So, if this represents a number, it cannot be rational. The issue of making mathematical sense of Plato's ideal was not resolved until the 19th century.

The main issues to be resolved are: what is number? what does it mean that a line consists of points? In the classical Greek days, line and point were undefined terms, and number was for counting. In the number line, number means a (signed) length: a is the signed length of the interval between a and 0 (positive is a is right of 0, negative if a is left of zero). Now, it was known by the Pythagoreans long before Plato that there exists a length (in the sense of *constructable by straightedge and compass* that is not representable by a natural number. Let us begin with the problem as it was understood at the time.

The discovery of the Pythagoreans is this: the diagonal and side of a square are *incommensurable*. Euclid (about a century after Plato) defined this term in this way. Take two intervals $A = [0, a]$ and $B = [0, b]$. A and B are commensurable if there is another interval R such that both A and B are integral multiples of R; that is, there are integers m and n such that $A = mR$ and $B = nR$. In particular, if B is the unit interval, $[0, 1]$, then A corresponds to the rational number m/n. If no such interval R exists, we say that A and B are incommensurable. As mentioned earlier, the Pythagoreans found out that the side and diagonal of a square are incommensurable; Euclid, in his *Elements*, gave a geometric proof of this fact, and Aristotle gave an arithmetic proof (both these will be further discussed in 8th grade).

Euclid based his argument on a test for commensurability that he invented, called the *Euclidean algorithm.* It goes like this: again, let A and B represent two (positive) lengths; let us suppose that $A > B$. Now we mimic the decimal expansion of A using B as the unit measure: start adjoining copies of B inside the interval A until we cannot add another copy without going beyond A.If we end up right at the end of A, we are done, for we have shown that A is an integral multiple of B. If not, let B_1 be the interval remaining after this fill. Since $B_1 < B$, we can start filling B with copies of B_1. If we end precisely at the endpoint of B, we are done: both A and B are integral multiples of

B_1. If not, let B_2 be the piece left over; since $B_2 < B_1$, we can try to fill B_1 with an integral multiple of B_2s, and so forth. If this process ends at any stage, the lengths A and B are commensurable; if not, they are incommensurable.

As a check on this procedure, we note that integer lengths are commensurable, since they are both integral multiples of the unit length. Thus, in this case, Euclid's test must end. And it does: when we put a copy of B inside A, since both A and B are integers, what remains is an integer also; in fact the integer $A - B$. So when we've put as many copies of B inside A as is possible, what remains B_1 is an integer less than B. Continuing this process, we get a decreasing sequence of integers, starting with B, so it must end. The remarkable thing is that the integer R at which this process ends is the greatest common divisor of A and B.

How could Euclid possibly show that the side S and the diagonal D are incommensurable; if he tried to keep repeating the process, he'd still be there in his little attic working away, and without a proof. Well: he was clever: he showed this: if we put the length S inside D, we have a piece P left over that is less than S. Now, he notices that P cannot fill S, and that the piece left over is smaller than P; but what is more, the geometry of the original square has been reproduced, only as a smaller version. So, since the geometry reproduces, the process can never end.

Let us give another example: that of the golden rectangle. A *golden rectangle* is a rectangle that is not a square, with this property: if we remove the square of whose side is the length of the smaller side of the rectangle, the remaining rectangle is a smaller version of the original. Clearly, if there is a golden rectangle, its sides cannot be commensurable. For, when we take away the square based on the smaller side, what remains is a reproduction of the original, at a smaller scale. This tells us that this process will never end.

Now, let's return to the representation of rational number p/q on the number line. We have been able to geometrically describe the arithmetic operations, but still, the sum of two rational numbers requires some explanation. Since the expression a/b is taken to mean a copies of $1/b$, then it is easy to add (or subtract) rational numbers with the same denominator: a copies of $1/b$ plus c copies of $1/b$ gives us $a + c$ copies of $1/b$:

$$\frac{a}{b} + \frac{c}{b} = \frac{1}{b}a + \frac{1}{b}c = \frac{1}{b}(a + c) = \frac{a + c}{b} .$$

Now, to add $a/b + c/d$ with $b \neq d$, we proceed as follows. First, divide the unit interval into bd equal pieces. Note that $1/b$ consists of the first d copies of $1/bd$, and $1/d$ consists of the first b copies of $1/bd$. In other words, the fractions $1/b$ and d/bd represent the same length, as do $1/d$ and b/bd. Thus:

$$\frac{a}{b} + \frac{c}{d} = a\frac{1}{b} + c\frac{1}{d} = a\frac{d}{bd} + c\frac{b}{bd} = \frac{ad}{bd} + \frac{cb}{bd} = \frac{ad + bc}{bd} .$$

Example 12.

$$-\frac{2}{5} + \frac{3}{7} = \frac{-14}{35} + \frac{15}{35} = \frac{-14 + 15}{35} = \frac{1}{35} .$$

Example 13.

How much is 2/5 of \$55.35?

Solution. There are several ways to do this. Here is one:

$$\frac{2}{5}(55 + \frac{35}{100}) = \frac{2}{5}55 + \frac{2}{5}\frac{35}{100} = 22 + \frac{14}{100} = 22.14 .$$

Let's take a second look at these computations. In the first fraction, divide 55 by 5 too get 11, and then multiply by 2 to get 22 (dollars). For the second fraction, divide 35 by 5 to get 7 and multiply 2 to get 14 (cents). Another way is to observe that $2/5 = 0.4$, and then multiply:

$$0.4 \times 55.35 = 0.4 \times 50 + 0.4 \times 5 + 0.4 \times .35 = 20 + 2 + 0.14 = 22.14 \,.$$

Solve real-world and mathematical problems involving the four operations with rational numbers.(7.NS.3)

Example 14.

Our group shared 3 pizzas for lunch. The pepperoni pizza was cut into 12 equally sized pieces, the tomato pizza into 8 pieces and the broccoli pizza into 6 pieces. I ate one piece of each. What fraction of a whole pizza did I eat?

Solution. My consumption was $1/12 + 1/8 + 1/6$. Now, I don't have to multiply $12 \times 8 \times 6$ since each denominator is a multiple of 24 : $24 = 2 \times 12 = 3 \times 8 = 4 \times 6$. Thus

$$\frac{1}{12} + \frac{1}{8} + \frac{1}{6} = \frac{2}{24} + \frac{3}{24} + \frac{4}{24} = \frac{9}{24} = \frac{3}{8} \,.$$

Example 15.

Genoeffa left her house and walked to the East down her street for a mile and a quarter. She then turned around and walked to the West for two and thee-eights miles. How far was she from her home?

Solution. We can think of her street as the number line, with her house at the zero position. Genoeffa walked in the positive direction to the point $1\frac{1}{4}$, and then walked in the negative direction a distance of $2\frac{3}{8}$. Using the number line model, her new position is at the point

$$1\frac{1}{4} + (-2\frac{3}{8}) \,.$$

Mixed fractions are often convenient in everyday discourse, but are very inefficient in a computation. So we rewrite the expression as

$$\frac{5}{4} - \frac{19}{8} \,.$$

Put both fractions over a common denominator, in this case 8, and subtract for the answer:

$$\frac{10}{8} - \frac{19}{8} = -\frac{9}{8} \,,$$

so Genoeffa ended up a mile and an eighth west of her house.

Chapter 3
Expressions and Equations Part 1

Making connections from concrete (specific / numeric) thinking to algebraic (involving unknown quantities / variables) thinking is a challenging but essential step in the mathematical progression of every student. Chapter 3 focuses on facilitating this transition by making connections to patterns (mathematical properties) already experienced for numbers and through repeated problems involving basic real-life examples. After developing understanding and procedural fluency with arithmetic properties of rational numbers, students learn how to manipulate equations to find solutions. Physical representations, including algebra tiles and area models, aid in understanding these operations for integer values and then extending them to include rational numbers. Using the distributive property "in reverse" helps students begin to master the important skill of factoring.

Once students have understood and achieved fluency with the algebraic processes, they then take real world situations, model them with algebraic equations, and use properties of arithmetic to solve them. The chapter concludes by having students model and solve percent increase and percent decrease problems that involve a little more algebraic thinking than the set of problems at the end of Chapter 1.

The objective of Chapter 3 is to facilitate students' transition from concrete representations and manipulations of arithmetic and algebraic thinking to abstract representations. Each section supports this transition by asking students to model problem situations, construct arguments, look for and make sense of structure, and reason abstractly as they explore various representations of situations. Throughout this chapter students work with fairly simple expressions and equations to build a strong intuitive understanding of structure. For example, students should understand the difference between $2x$ and x^2 or why $3(2x?1)$ is equivalent (equal in value) to $6x - 3$ and $6x + (?3)$. Students will continue to practice skills manipulating algebraic expressions and equations throughout Chapters 4 and 5. In Chapter 6, students will revisit ideas in this chapter to extend to more complicated contexts and manipulate with less reliance on concrete models.

Another major theme throughout this chapter is the identification and use in argument of the arithmetic properties. The goal is for students to understand that they have used the commutative, associative, additive and multiplicative inverse, and distributive properties informally throughout their education. They are merely naming and more formally defining them now for use in justification of mathematical (quantitative) arguments.

In essence, Chapter 3 is an extension of skills learned for operations with whole numbers, integers and rational numbers to algebraic expressions in a variety of ways. For example, in elementary school students modeled 4×5 as four "jumps" of five on a number line. They should connect this thinking to the meaning of $4x$ or $4(x + 1)$. Students also modeled multiplication of whole numbers using arrays in earlier grades. In this chapter they will use that logic to multiply using unknowns. Additionally, in previous grades, students explored and solidified the idea that when adding/subtracting one must have *like units*. Thus, when adding 123 + 14, we add the "ones" with the "ones," the "tens" with the "tens" and the "hundreds" with the "hundreds." Similarly, we cannot add 1/2 and 2/3 without a common denominator because the unit of $1/2$ is not the same as a unit of $1/3$. Students should extend this idea to adding variables. In other words, $2x + 3x$ is $5x$ because the unit is x, but $3x + 2y$ cannot be simplified further because the units are not the same (three of x and two of y). This should be viewed as a situation similar to that posed by: $2/3 + 4/5$, which cannot be simplified because the units are not the same (but here we have fraction equivalence to replace both fractions by equivalent fractions with the same denominator.

In 6th grade, students solved one-step equations. Students will use those skills to solve equations with multiple steps in this chapter. Earlier in this course, students developed skills with rational number operations. In this chapter, students will be using those skills to solve equations that include rational numbers.

Section 3.1 reviews and builds on students' skills with arithmetic from previous courses, as previously noted, to write basic numeric and algebraic expressions in various ways. In this section students work on understanding the difference between an expression and an equation. Further, they should understand how to represent an unknown in either an expression or equation. Students will connect manipulations with numeric expressions to manipulations with algebraic expressions. In connecting the way arithmetic works with integers to working with algebraic expressions, students name and formalize the properties of arithmetic. By the end of this section students should be proficient at simplifying expressions and justifying their work with properties of arithmetic.

Section 3.2 uses the skills developed in the previous section to solve equations. Students will need to distribute and combine like terms to solve equations. In Grade 7, students only solve linear equations in the form of $ax + b = c$ or $a(x+b) = c$, where a, b, and c are rational numbers. This section will rely heavily on the use of models to solve equations, but students are encouraged to move to abstract representation when they are ready and fluent with the concrete models.

We close the chapter with section 3.3 which is about converting contextual (story) problems into algebraic equations and solving them. Contexts involve simple equations with rational numbers so the focus will be on concept formation and abstraction (solving algebraic formulations). Percent increase and decrease is revisited here. Time will be spent understanding the meaning of each part of equations and how the equation is related to the problem context. Note that the use of models is to develop an intuitive understanding and to transition students to abstract representations of thinking.

As students move on in this course, they will continue to use their skills in working with expressions and equations in more complicated situations. The idea of inverse operations will be extended in later grades to inverse functions of various types. A strong foundation in simplifying expressions and solving equations is fundamental to later grades. Students will also need to be proficient at translating contexts to algebraic expressions and equations and at looking at expressions and equations and making sense of them relative to contexts.

One of the fundamental uses of mathematics is to model real-world problems and find solutions using valid steps. Very often, this involves determining the value for an unknown quantity, or unknown. In Grade 8 we extend this concept to include that of a variable, a representation for a quantity that can take on multiple values. In this chapter we want to work on making several important distinctions. First is that between expressions and equations. The analogy is with language: the analog of "sentence" is equation and that of "phrase" is expression. An equation is a specific kind of sentence: it expresses the equality between two expressions. Similarly, an inequality makes the statement that one expression is greater than (or less than) another; a topic that will be further developed in Chapter 6. We note that statements can be true or false (or meaningless). In fact, the problem to be dealt with in this chapter is to discover under what conditions an equation is true; that is, what is meant by solving the equation?

These equations involve certain specific numbers and letters. We refer to the letters as unknowns, that is they represent actual numbers, but they are not yet made specific; our task is to do so. If an equation is true for all possible numerical values of the unknowns (such as $x + x = 2x$), then the equation is said to be an *equivalence*. Arithmetic operations transform expressions into equivalent expressions. Simplifying an expression or equation is a similar process of applying arithmetic properties to an expression or equation, but, in the case of an equation, may not go so far as to find the solution. Why are we interested in simplifying? When we simplify we make things "simpler" or reduce the number of symbols used while retaining equality, that is, to make the expression as short and compact as possible. Ultimately we are interested in finding that (or those, if any) numbers which when substituted for the unknown make the equation true. These are called the solutions.

3.1: Communicate Numeric Ideas and Contexts Using Mathematical Expressions and Equations

This section contains a brief review of numeric expressions, with recognition that a variety of expressions can represent the same situation. Models are utilized to help students connect properties of arithmetic in working with numeric expressions to working with algebraic expressions. These models, particularly algebra tiles, aid students in the transition to abstract thinking and representation. Students extend knowledge of mathematical properties (commutative property, associative property, etc.) from purely numeric problems to expressions and equations. The distributive property is emphasized and factoring, "backwards distribution," is introduced. Work on naming and formally defining properties appears at the beginning of the section so that students can attend to precision as they verbalize their thinking when working with expressions. Throughout the section, students are encouraged to explain their logic and critique the logic of others.

Algebraic thinking doesn't just begin in Grade 7; it was started in the very early grades, in the sense that mathematics is about solving problems, and algebra provides the necessary algorithms, as in problems like "$3 + \_\_ = 5$."

Early algebraic problems such as this are first solved using concrete manipulatives. However, as students develop their mathematical reasoning, they begin to use more abstract representational processes. Patterns in solving similar problems lead to both an understanding of, and familiarity with, arithmetic properties.

Apply properties of operations as strategies to add, subtract, factor, and expand linear expressions with rational coefficients. 7.EE.1

Students begin this section with a review of numeric examples and then extend that understanding to the use of unknowns. The following example gives a flavor of the progression of thinking.

EXAMPLE 1.

For the following problems, a soda costs \$1.25 and a bag of chips costs \$1.75.

- **a.** Mary bought a soda and a bag of chips. How much did she spend?
- **b.** Viviana bought 3 sodas and 2 bags of chips. How much did she spend?
- **c.** Martin bought 2 sodas and some bags of chips, and spent a total of \$7.75. How many bags of chips did Martin buy?
- **d.** Paul bought s sodas and 4 bags of chips. Write an expression for how much Paul spent.

SOLUTION.

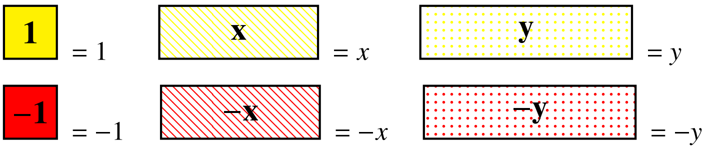
- **a.** \$1.25 + \$1.75 = \$3.00.
- **b.** Viviana bought 3 sodas at \$1.25 each, so spent 3(\$1.25) = \$3.75 on soda. She bought 2 bags of chips at \$1.75 a bag, so spent \$3.50 on chips. All together she spent \$7.25.
- **c.** Martin spent \$2.50 on his sodas, and all the rest on chips. So, he spent \$5.25 on chips. Since the cost of chips is \$1.75 a bag, he bought \$5.25/\$1.75 = 3 bags of chips.
- **d.** At \$1.25 each, the cost of s sodas is \$1.25(s) ; at \$ 1.75 each, the cost of 4 bags of chips is \$7.00. So, Paul spent \$1.25s + \$7.00.

One of the primary goals of this section is to help students recognize that properties of arithmetic, the rules they've informally observed throughout their formal and informal education that govern whole numbers and fractions, extend to all quantities (integers, rational numbers, and unknown quantities represented by letters or symbols). As the chapter unfolds, students will begin to name these properties, but first students must understand that addition and multiplication with unknown values "work" the same as they do with known values. For example, in addition, when calculating $(3x+2)+(4x+6)$, students have learned that they can't simply add up all the digits $(3+2+4+6 = 15)$. Instead, the place value involved in the notation tells students that there are three groups of x and two units in the first set and four groups of x and six units in the second set. From previous experience (with numeric place value), they know that to add you need to join the three x's and the four x's together because they are "alike" (based on the same size pieces). Also, the two units and six units are put together because they are units of the same value. Therefore, the actual answer is seven x's and eight units, written $7x + 8$.

Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations to solve problems by reasoning about the quantities. 7.EE.4

To help make the desired connections between concrete numbers and abstract algebraic expressions, manipulatives similar to base-ten blocks, called algebra tiles, are introduced. These algebra tiles aid in understanding the processes of addition, subtraction, multiplication, and combining like terms because constants (positive and negative) and variables (positive and negative) each have a distinct shape. This commonality of shape encourages students to group the appropriate terms together, helping to avoid common mistakes. Although algebra tiles are for expressions with integer coefficients, the process extends to rational numbers. Early examples involve only one variable but later examples involve multiple variables, as shown below.

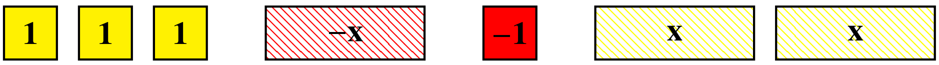
Algebra Tile Key:



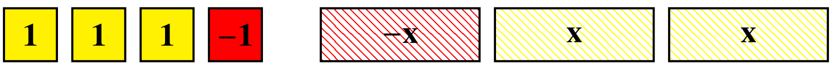
Example 2.

Model the expression $3 - x - 1 + 2x$ and then simplify, combining like terms.

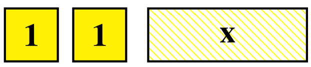
Solution. First model each component of the expression with the corresponding algebra tile(s).



As with the base-ten blocks used in earlier grades, next put like terms together



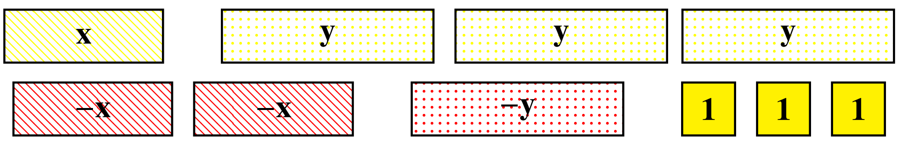
Since $1 + -1 = 0$ and $x + -x = 0$, we have left the expression $2 + x$ (or, equivalently, $x + 2$).



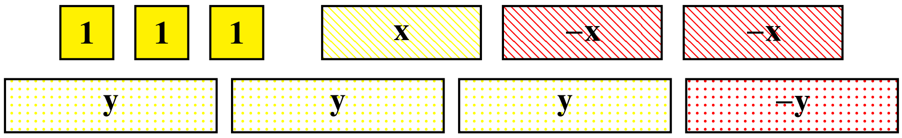
Example 3.

Model the expression $x + 3y - 2x + -y + 3$ and then simplify.

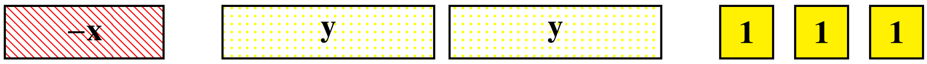
Solution. First lay out the corresponding algebra tiles.



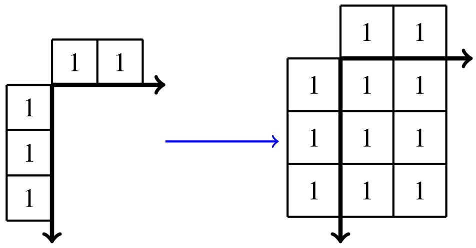
Next, combine like terms by grouping tiles of the same shape.



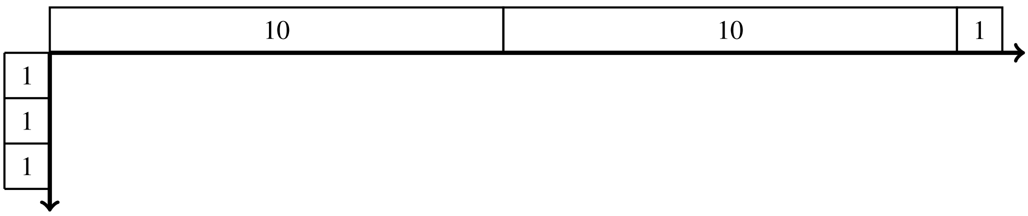
Since $1 + (-1) = 0$, $x + (-x) = 0$, and $y + (-y) = 0$, we eliminate some tiles, giving $-x + 2y + 3$.



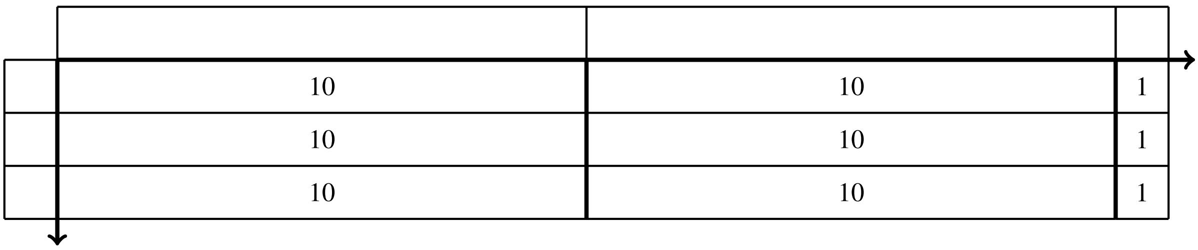
After simplifying expressions, students then move to iterating groups and the distributive property as other ways to view the same expression. The understanding for this is built off of the area model for multiplication learned in earlier grades. Students have already learned that $3 \cdot 2$ can be represented as three rows of two, yielding a product (inside area of six).



Similarly, we can use base-ten blocks to understand the meaning of $3 \cdot 21$ as three groups of the quantity two tens and one unit. In previous grades, students have written this out as $3 \cdot (20 + 1)$.



Once modeled, the students see that the inside area is six tens and three units or 63.



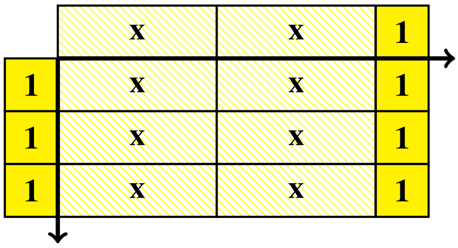
This naturally leads students to extend the knowledge to related algebraic expressions, for example $3 \cdot (2x + 1)$. Again the algebra tiles help make the desired connections, as in the following problem.

Example 4.

Model the expression $3 \cdot (2x + 1)$ and then simplify.

Solution. Use the area model but include variable tiles.

We see that the product (inside area) consists of 6 x-tiles and 3 1-tiles, so another way to write $3 \cdot (2x + 1)$



is $6x + 3$. This is perfectly analogous to the previous example except that we have used x-tiles in place of "base-ten" blocks.

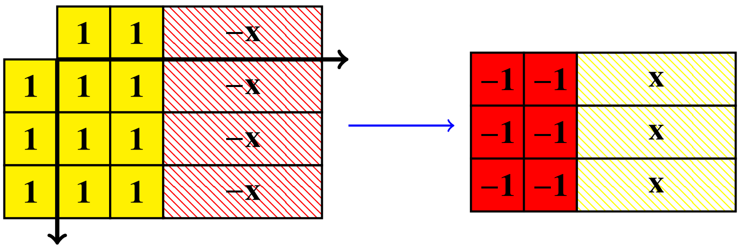
Only one more step is needed when modeling a problem where an expression is multiplied by a negative integer. Emphasis needs to be placed on "$-$" meaning "the opposite": $2 + (-2) = 0$, $3 + (-3) = 0$, and in fact, for any number a, $a + (-a) = 0$. Subtraction can be viewed as adding the opposite: $a - b = a + (-b)$. All this may take time to unravel, say for an expression like $a - (-b)$. This should be read as $a + -(-b)$: in words, a plus the opposite of $-b$. Since the opposite of $-b$ is b, this becomes $a + b$. So, $5 - (3 - 4) = 5 + (-(3 - 4)) = 5 + (4 - 3)$ (since the opposite of $3 - 4$ is $4 - 3$, and so the result is $5 - (3 - 4) = 6$. This is not easy to understand without parsing the phrase $5 - (3 - 4)$ just as we parse a phrase in language.

Recall from the discussion in the preceding chapter, that, for whole numbers a and b, $a(-b)$ is considered as a groups of $-b$. Note that this is the same as the opposite of a groups of b : $a(-b) = -(a \cdot b)$. Finally, the extension of the laws of arithmetic leads to $-(a?b) = (-a)b$.

Example 5.

Model and simplify $-3 \cdot (2 - x)$.

Solution. Realizing that -3 is the opposite of 3, first model $3 \cdot (2 - x)$ with algebra tiles. Recall that $2 - x$ is the same as $2 + -x$.



The inside area is $6 - 3x$. But this was the model for $3 \cdot (2 - x)$ and the original problem was $-3 \cdot (2 - x)$. Therefore, we need to take the opposite of all the inside tiles, getting $-6 + 3x$.

Students should recognize this as a model for the distributive law.

We continue to emphasize that the same patterns that held for numeric expressions also hold for algebraic expressions. At this point, students are introduced to the names of these properties and they practice using them as justification for each step in solving mathematical problems. It is also important to note that these same properties continue to hold as we extend to the real numbers in Grade 8 and complex numbers in Secondary II. In more advanced classes, systems that have the same properties will be given the name of field and will behave in similar ways. The table below summarizes these properties, making explicit that "iterating groups" is the distributive property.

Property Name	**Concrete Example**	**General Equation**
Identity Property of Addition	$7/10 + 0 = 7/10 = 0 + 7/10$	$a + 0 = a = 0 + a$
Identity Property of Multiplication	$9.04 \cdot 1 = 9.04 = 1 \cdot 9.04$	$a \cdot 1 = a = 1 \cdot a$
Multiplicative Property of Zero	$5/8 \cdot 0 = 0 = 0 \cdot 5/8$	$a \cdot 0 = 0 = 0 \cdot a$
Commutative Property of Addition	$5.8 + 2.4 = 2.4 + 5.8$	$a + b = b + a$
Commutative Property of Multiplication	$3.6 \cdot 7.1 = 7.1 \cdot 3.6$	$a \cdot b = b \cdot a$
Associative Property of Addition	$1/2+(3/5+1/4) = (1/2+3/5)+1/4$	$a+(b+c) = (a+b)+c$
Associative Property of Multiplication	$2 \cdot (3.1 \cdot 7.5) = (2 \cdot 3.1) \cdot 7.5$	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
Distributive Property of Addition over Multiplication	$2.8 \cdot (6.1+4.35) = 2.8 \cdot 6.1+2.8 \cdot 4.35$	$a \cdot (b+c) = a \cdot b + a \cdot c$
Additive Inverse Property	$7/16 + (-7)/16 = 0$	$a + (-a) = 0$
Multiplicative Inverse Property	$3/25 \cdot 25/3 = 1$	$a \cdot 1/a = 1$, for $a \neq 0$

Students practice applying these properties as they step through the simplification process logically. This is shown in the next example. It is essential to concentrate on understanding, and not vocabulary. It does no good to be able to illustrate or define any one of these properties without understanding them.

Example 6.

Consider the expression $2(3x + 1) + -6x + 3$. In the first column of the following table are various expressions. Some can be derived from the given expression , and others cannot, In the second column, if the expression is derivable from the given expression, give the relevant instructions. In the third column, justify the instruction by the appropriate property. If the expression cannot be derived form the given expression, just write *No* in the second column.

	Expression	Steps	Justification
a.	$2(3x + 1) + -6x + 3$	No change	Given expression
b.	$6x - 3$	No	
c.	0	Yes	Multiplicative property of 0
d.	$6x + 2 + -6x + 3$		
e.	$6x + 2 + -6x - 3$		
f.	$6x + -6x + 2 + 3$		
g.	$6x + 1 - 6x + 3$		
h.	$0 + 5$		
i.	5		

Solution.

	Expression	Steps	Justification
a.	$2(3x + 1) + -6x + 3$	No change	Given expression
b.	$6x - 3$	No	
c.	Multiply by 0	Yes	Multiplicative property of 0
d.	$6x + 2 + -6x + 3$	Multiplied both $3x$ and 1 by 2	Distributive Property
e.	$6x + 2 + -6x - 3$	No	
f.	$6x + -6x + 2 + 3$	Change the order of the terms	Commutative Property of Addition
g.	$6x + 1 - 6x + 3$	No	
h.	$0 + 5$	From f., $6x + (-6x) = 0$	Additive Inverse
i.	5	Delete 0+ from h	Additive identity

Example 7.

1. In the above problem, in each case explain, in each case of an entry of "No" in the second column, how you arrived at that conclusion.

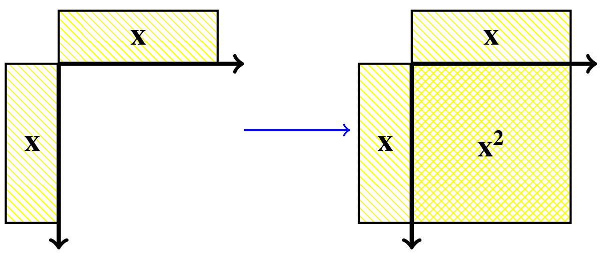
2. In each case, can the old expression be derived from the new one?

Solution.

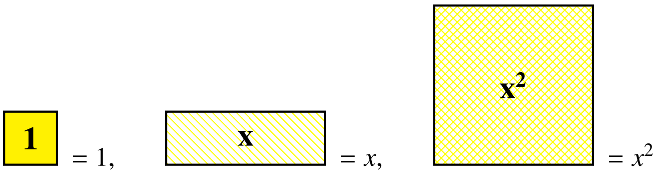
Part 1. This is a critical problem, exploring what the student thinks is going on when these rules are applied. Correct responses will vary greatly, but, to be correct, they should center around the idea that true statements remain true after an application of a "law of arithmetic." So, the response to b. is "No" for this reason: if we substitute 0 in the given equation, we get 5, but the same substitution in b. gives us -3. So the true statement $(2(3(0) + 1) + -6(0) + 3 = 5$ transforms into the false statement $6(0) - 3 = 5$. No law of arithmetic can allow that.

As for Part 2 , the answers, in each case, reduce to this: can the effect of the laws applied be undone by application of these laws? Except for multiplication by zero, all the laws can be undone, For example, the distributive law takes us from $5(3 + 7)$ to $5(3) + 5(7)$, and that brings us to $15 + 35$, which is 50. That can be turned around: 50 can be rewritten as $5 \cdot 10$, and $10 = 3 + 7$, so 50 can be written as $5 \cdot (3 + 7)$. The only law that cannot be undone is multiplication by 0.

We conclude this section by looking at the distributive property both "forward" and "in reverse." Recalling the area model for multiplication, students understand why $x \cdot x$ is called "x-squared" and expand the algebra tile key.



Key:

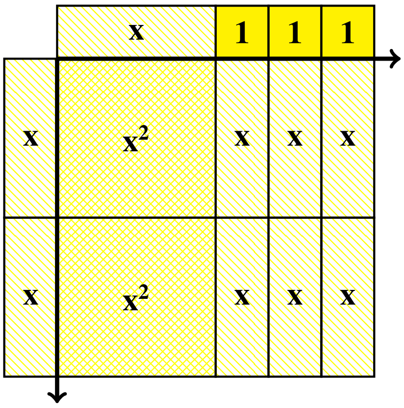


Combined with previous knowledge, students can now see the connection between arithmetic properties and expressions involving variables, especially multiplication and the distributive law. This is an important step as the student progresses from concrete to abstract.

Example 8.

Use algebra tiles to find $2x \cdot (x+3)$. Then write an equation of how this models the distributive property.

Solution. Use the factors of $2x$ and $x + 3$ and the area model.



There are 2 x^2 pieces and 6 x pieces for an area of $2x^2 + 6x$. Therefore,

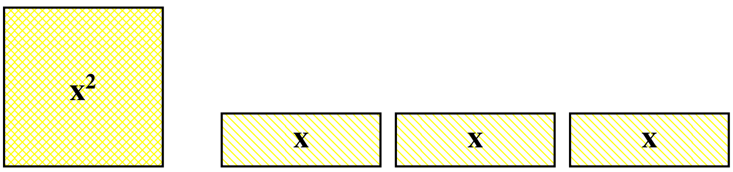
$$2x \cdot (x+3) = 2x \cdot x + 2x \cdot 3 = 2x^2 + 6x\,.$$

Reversing the above process allows students to use the distributive property backwards, a process called factoring that will be used heavily in later courses. This is seen in the following example.

Example 9.

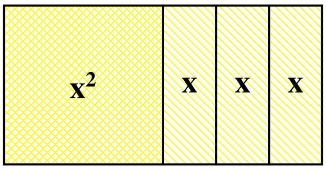
Use the area model to factor the expression $x^2 + 3x$.

Solution. Start with the given information, namely one tile for x^2 and three x pieces.

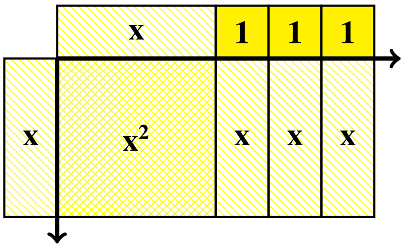


Since the area model for multiplication makes a rectangle, combine the pieces in such a way as to make one large rectangle. One method is shown below.

Looking at this as a multiplication in reverse, we see that this would be x down the left and x plus three



1s across the top. So the factors of $x^2 + 3x$ are x and $x + 3$. The equation showing the factorization (and connecting to the distributive property) would be $x^2 + 3x = x \cdot (x + 3)$.



Section 3.2: Solve Multi-Step EquationsSolving Algebraic Equations (ax + by = c)

Having achieved proficiency at using arithmetic properties to manipulate expressions, students are now ready to take the next step and apply their knowledge to solving equations. Recall that an equation is an assertion that two expressions are equal. In Grade 7 an algebraic equation will assert the equality between two expressions involving an unknown represented by a letter, usually x, but often, in context, with a letter that suggests the specific unknown to be determined. For example, if one is given information to use to discover Darren's age, we might use the symbol d to represent Darren's age. If the given information is that two years ago Darren was 11 years old, we can write this as the algebraic equation $d?2 = 11$. The object of this section is to have students understand and use the laws of arithmetic in order to solve these equations for the unknown.

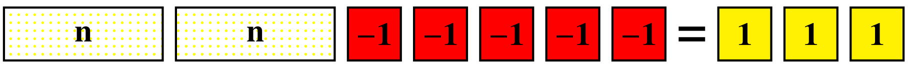
Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. 7.EE.3

Students begin the section by modeling and solving equations with integer coefficients. Patterns in solving similar problems with models (algebra tiles) lead to solution methods that do not require diagrams. This then allows the process to extend to problems with rational numbers. Also, students will reinforce the properties learned in the previous section as they justify each step in the solution process.

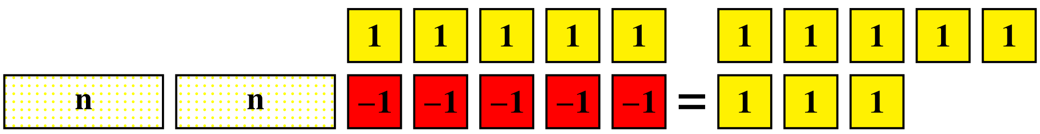
Example 10.

. Solve $2n - 5 = 3$.

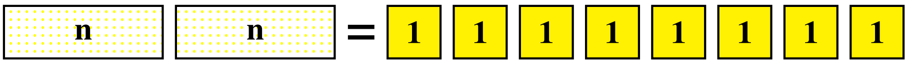
Solution. (using algebra tiles). Model with tiles:

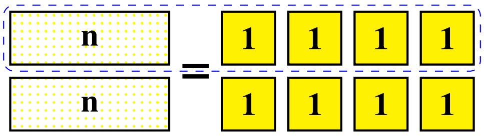


Add the same amount, 5, to both sides. This is legitimate since the values of n that make the above equation true are precisely the same values of n making the next statement true.



Using $1 + -1 = 0$, the additive inverse property, combine like terms. Rearrange tiles and the solution is 4.





Solution. (without algebra tiles).

Given $2n - 5 = 3$, add 5 to each side (addition property of equality) of the equation: $2n - 5 + 5 = 3 + 5$.

By the additive inverse property, $-5 + 5 = 0$ so $2n + 0 = 8$.

Using the additive identity property, $2n = 8$.

Dividing both sides of the equation by 2 gives $n = 4$.

When arriving at a result, it is always good practice to verify that it is the desired value. Substituting $n = 4$ into the original equation, we see that $2(4) - 5 = 3$. A little arithmetic shows that the equation is true when n = 4 so we have checked that the result is in fact the solution to the equation.

Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations to solve problems by reasoning about the quantities. 7.EE.4.

After achieving procedural fluency, students then model real world situations by equations and find corresponding solutions as illustrated in the example below.

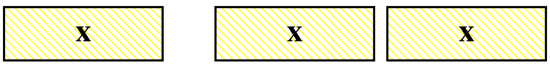
Example 11.

Bill has twice as much money as I do. Together we have \$18.45. How much money do we each have?

Solution. (with a model).

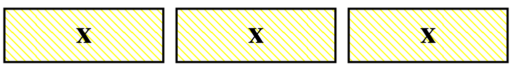
Since we do not know how much money I have, let's assign it an unknown, x. This can be represented by an x algebra tile. Bill has twice as much money as I have which would be modeled by two x algebra tiles, as is illustrated below.

My money Bill's money (twice my money)

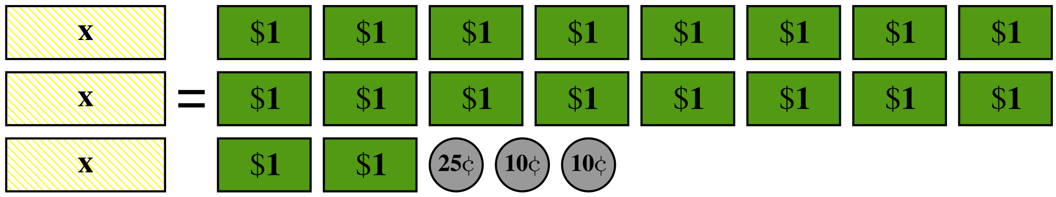


Since the problem tells us how much money we have together, we add our algebra tiles (join our resources). This yields$x + (x + x)$ or $3x$.

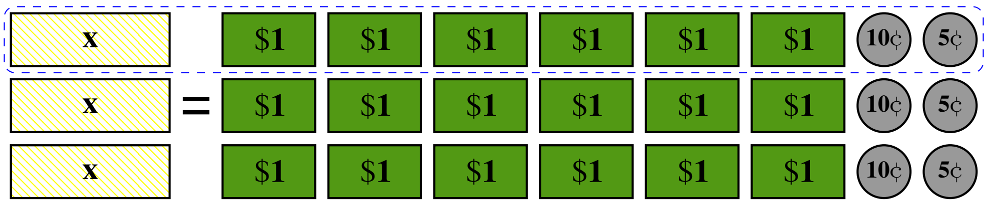
Our money together



Since our money together is represented by $3x$ and that is equal to the amount \$18.45, we have the following model.



Trading the quarter for a dime and three nickels and then rearranging into equal groups next to each x-tile, we see that $1x = x = \$6.15$.



Therefore, I have \$6.15 and Bill has double that amount, \$12.30. Adding these amounts together gives \$18.45 so we have verified that this is the correct answer.

Solution. (without a model). Let x be the amount of money I have. Then, since Bill has twice as much money, $2x$ is the amount of money he has.

Putting the amounts together leads to the equation $x + 2x = \$18.45$.

Combining like terms gives $3x = \$18.45$.

Dividing both sides by 3 yields the equation $x = \$6.15$.

Therefore, I have \$6.15 and Bill has twice that amount $2 \cdot \$6.15 = \$12.30$.

Again, a quick check shows that \$6.15 + \$12.30 equals \$18.45 so we have verified that our result is the solution to the equation.

Section 3.3: Solve Multi-Step Real-World Problems Involving Equations and Percentages

This chapter concludes with the modeling of, and finding the solution to, common real-world percent problems. We begin with a quick review of the content of Chapter 1, Section 3. Percent increase problems arise in everyday scenarios such as price plus sales tax, tip on a meal, and pay raises. Percent decrease problems occur when dealing with discounts or similar reduction scenarios. Students use a diagram to model the situation, convert the model into an expression or equation, and then simplify or solve using the skills recently developed. Models are emphasized in this section to help students understand the structure of the problems and they will rely on more algebraic reasoning later in the course. Note that if the problem involves an x percent increase of a quantity q then the result is $q + (x/100)q$ which can also be written as $q(1 + x/100)$. Similarly, for an x percent decrease problem, the expression will be $q?(x/100)q$ or $q(1?x/100)$.

Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related. For example, $a + 0.05a = 1.05a$ means that "increase by 5% " is the same as "multiply by 1.05" 7.EE.2

Example 12.

Lucia takes some friends out to lunch and the bill comes to \$b. She wants to leave a 20% tip for the waitress. How much money does she need for the food and tip together?

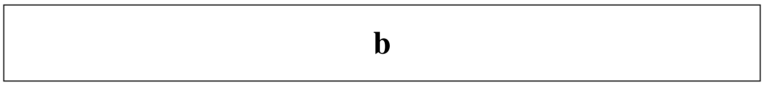
a. Draw a model to represent the situation.

b. Write at least two equivalent expressions to represent the total amount Lucia pays.

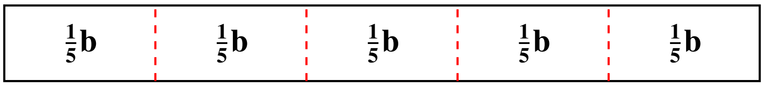
c. If the cost of the food was \$28.50, how much will Lucia need to pay?

Solution.

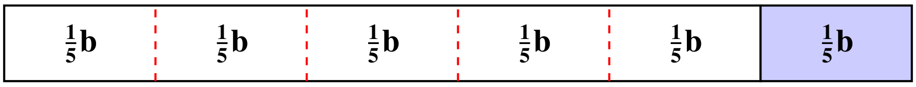
a. Since b is the amount of the lunch bill, draw a rectangle to represent it.



Because 20% is equal to one-fifth, divide the original rectangle into five equal parts.



Since the tip is in addition to the bill, Lucia will have to pay an extra one of those five parts.



b This diagram leads to the expressions $b + (1/5)b$ and $b + 0.2b$. Another way to express this is that Lucia will pay for 6/5 or 1.2 times the original price, leading to the expressions $(6/5)b$and $1.2b$.

c. If the food cost \$28.50, then the food and tip together will cost $1.2 \cdot \$28.50 = \$34.20$.

Example 13.

Jordan wants to buy a soccer ball that costs s dollars. Today, all sports items are on sale for 25% off regular price.

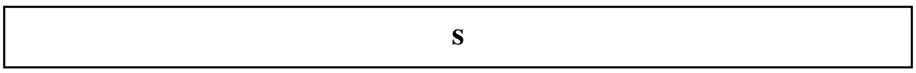
a. Draw a model to represent the situation.

b. Write at least two equivalent expressions to represent the sale price of the ball.

c. If the original price of the ball was \$12.96, what is the sale price of the ball?

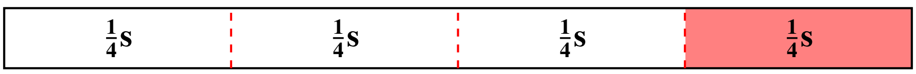
Solution.

a. Since s is the original price of the ball, draw a rectangle to represent original price.



Since 25% is equal to one-fourth, divide the rectangle into four equal parts.

Since the ball is discounted 25%, Jordan will not have to pay for one of those 4 parts.



b. This diagram leads to the expression $1s - 0.25s$.

Another way to express this is that Jordan will pay for three-fourths of the original price, leading to the expression $3/4s$.

Note that $1s - 0.25s = 0.75s = 3/4s$.

c. Given that $s = \$12.96$, the sale price of the ball would be $0.75¢ \cdot \$12.96$ which has a value of \$9.72.

Solve word problems leading to equations of the form $px + q = r$ and $p(x + q) = r$, where p, q, and r are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of operations used in each approach. For example, the perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width? 7.EE.4a

Example 14.

Drake wants to buy a new skateboard with original price of s dollars. The skateboard is on sale for 20% off the regular price; the sale price is \$109.60. Find the original price of the skateboard.

Solution.

$$s - 0.2s = 109.60 \quad \text{or} \quad 0.8s = 109.60$$

from which we find that $s = 137$: the original price of the skateboard was \$137.

The goal here is to understand structure of percent equations, that is, recognizing that the 20% off an original amount is the same as 80% of the original amount.

Example 15.

Drake decided to buy the skate board for the sale price. When he got to the counter, he got an additional 10% off, but also had to pay 8.75% sales tax. What was the total amount Drake paid for the skateboard?

Solution. The additional cut of 10% means he paid, 90% of \$109.60, but then had 8.75% of that amount added in tax:

$$0.90(109.60) + 0.0875(0.90(109.60)) = 98.64 + 8.63 = 107.27 \ .$$

Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. 7.EE.3

As students start to solve word problems algebraically, students also use more complex expressions, eliciting guidance in order to develop the strategy of working forwards, rather than backwards. In essence, working with numeric expressions prepares students for work with algebraic expressions.

Example 16.

Bruno earns \$43,250 a year. Of this amount, he pays 17.8% to taxes. Of the remainder, 1/3 is for living expenses, 2/5 for food and entertainment, and 1/4 for other insurance and car expenses. What percent of the \$43,250 does Bruno have left over for miscellaneous expenses? How much money is left over?

Solution. Bruno's taxes are $0.178 \times 43250 = 7698.5$ dollars. So, his after-tax income is $43250 - 7698.5 = 35,551.50$. Of that 1/3+2/5+1/4 is allocated to recurring expenses. Now

$$\frac{1}{3} + \frac{2}{5} + \frac{1}{4} = \frac{20}{60} + \frac{24}{60} + \frac{15}{60} = \frac{59}{60}\ .$$

So what remains is 1/60 of his after-tax income, or $(1/60) \times 35551.50 = 592.53$. Since 592.53/43250 = 0.0137, we can conclude that 1.37% of his total pay is available for misc. expenses, and consequently, all work and no play makes Bruno a dull boy.

Chapter 4
Analyze Proportional Relationships and Use Them to Solve Real-World Problems

Because of their prevalence and importance in science and every-day life, it is important that students understand ratios and proportional relationships so they become informed consumers and critical thinkers. This chapter focuses on that understanding starting with the concept of unit rate as introduced in sixth grade. Two related quantities A and B are said to be in the *ratio relationship* $a : b$ if the number of elements in A, a, divided by the number of elements in B, b, is always the same value. This is often written as $a : b$. The quotient a/b is the **unit rate** of A with respect to B, meaning for every unit of B there are a/b units of A. For example, if I can walk 2 laps around the track every 10 minutes, the unit rate is 2/10 = 1/5 lap per minute.

In 7th grade we introduce the concept of a *proportional relationship* between two quantities, and relate it to the idea of ratio. In order to develop a broad understanding, we use a variety of approaches, including bar models, tables, graphs, and equations. Students will use these tools to determine when two quantities are proportional. Working with proportional relationships allows one to solve many real-life problems such as adjusting a recipe, quantifying chance (odds and probability), scaling a diagram (drafting and architecture), and finding percent increase or percent decrease (price markup, discount, and tips). The chapter begins with an anchor problem about lemonade recipes which raises important ideas of unit rates, ratio and proportion.

The study of ratios and proportional relationships extends students' work with multiplication, division and measurement from earlier grades and forms a strong foundation for further study in mathematics and science. Later in this course, students will use proportions to solve scaling problems, including making scale drawings. In 8th grade, proportions form the basis for understanding the concept of constant rate of change (slope) and students learn that proportional relationships are a subset of linear relationships, and thus are represented by linear functions. As students progress, they use ratios in algebra (functions), trigonometry (the basic trigonometric functions) and calculus (average and instantaneous rate of change of a function). An understanding of ratio is essential in the sciences to make sense of speed, acceleration, density, surface tension, electric or magnetic field strength, and strength of chemical solutions. Ratios also appear in descriptive statistics, including demographic, economic, medical, meteorological, and agricultural statistics (e.g. birth rate, per capita income, body mass index, rain fall, and crop yield). Ratios underlie a variety of measures, for example, in finance (exchange rate), medicine (dose for a given body weight), and technology (kilobits per second).

Section 4.1: Understand and Apply Unit Rates

Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units. 7.RP.1

In 6th grade, students learned that a ratio is a comparison of the size of two quantities. An example is the relation between the number of boys and girls in the class. One possible instance would be a class of 28 students composed of 16 girls and 12 boys. The ratio could be looked at in several ways: part-to-part as in 16 girls : 12 boys or part-to-whole as in 16 girls : 28 students. This section will briefly review concepts from 6th grade and extend them to proportions and unit rates with the goal of solving real-world problems. Students connect and build on this knowledge by first modeling situations with tape diagrams or bar models, illustrated in the first examples.

EXAMPLE 1.

A chocolate chip cookie recipe calls for three cups of flour and two cups of sugar. Use a tape diagram (bar model) to represent this situation.

SOLUTION.

1 cup flour	1 cup flour	1 cup flour	1 cup sugar	1 cup sugar

EXAMPLE 2.

If Alex only has one cup of sugar, how much flour should he use in making the cookies?

SOLUTION. Alex notices that he has half the sugar needed for the recipe, and therefore can only make half a recipe. He copies the above tape, but changes units to half what they were in example 1.

1/2 cup flour	1/2 cup flour	1/2 cup flour	1/2 cup sugar	1/2 cup sugar

Now Alex observes that this formulation shows the amount of sugar he has available, and so he will need 3 half cups of flour; that is 1.5 cups of flour.

EXAMPLE 3.

Alex wants to make many cookies to bring to a party so he buys more sugar. If he uses six cups of sugar how much flour will he need?

SOLUTION. Following the same logic, Alex can repeat the first tape above but now the units are "3 cups." Alternatively he can reason that he will now have 3 times the sugar needed for the first tape recipe, so all he has to do is repeat that tape 3 times.

1 cup flour	1 cup flour	1 cup flour	1 cup sugar	1 cup sugar
1 cup flour	1 cup flour	1 cup flour	1 cup sugar	1 cup sugar
1 cup flour	1 cup flour	1 cup flour	1 cup sugar	1 cup sugar

Example 4.

Riley has a job helping the neighbor with yardwork. One day Riley worked for two hours and earned \$11.00. How much does she earn each hour?

Solution. Unlike cups of flour and cups of sugar, dollars and hours are different units of measure. We create a comparison tape diagram by drawing two bars of equal length, one on top of the other, to represent the situation. We model each quantity in its own bar.

2 hours
\$11.00

Since we are looking for the amount Riley makes each hour, and there are two hours, divide each bar into two equal pieces. This gives 1 hour in each piece of the top bar and \$5.50 in each of the bottom bars. Therefore, Riley makes \$5.50 per hour helping the neighbor.

1 hour	1 hour
\$5.50	\$5.50

1 hour						1 hour				
\$ 1	\$1	\$1	\$ 1	\$1	\$1	\$ 1	\$1	\$1	\$1	\$1

Example 5.

Riley wants to make a table of how much money she'll earn by working 0, 1, 2, 3, 4, and 5 hours.
Solution. The tape diagram can again be iterated to model the situation.

1 hour	1 hour	1 hour	1 hour	1 hour
\$5.50	\$5.50	\$5.50	\$5.50	\$5.50

Hours	0	1	2	3	4	5
Dollars	0	5.50	11.00	16.50	22.00	27.50

Notice the patterns in the table from the previous example. Each time the number of hours increases by one, the number of dollars increases by 5.50. If you double the number of hours worked from two to four, the amount of money earned also doubles from \$11.00 to \$22.00. Also note that the number of dollars is equal to the number of hours multiplied by 5.50. Often it is the case that different instances of two quantities lead to **equivalent ratios**.

Two ratios $a : b$ and $c : d$ are **equivalent** if there is a number p such that $c = ap$ and $d = bp$. If the ratios are considered in fractional form, the ratio $a : b$ is equivalent to the ratio $c : d$ if $\frac{c}{d} = \frac{ap}{bp}$ where denominators are nonzero.

Note that the notion of equivalence of ratios is the same as the equivalence of fractions, and the fraction is the unit rate: $a : b$ is equivalent as a ratio to $c : d$ precisely when the fractions a/b and c/d are equivalent.

In the cookie examples, the ratios of flour to sugar are expressed in the different scenarios as 3:2, (3/2):1, and 9:6. Since $3 = (3/2) \cdot 2$, $2 = 1 \cdot 2$, $9 = (3/2) \cdot 6$, and $6 = 1 \cdot 6$, the ratios are equivalent. Similarly, in Riley's example, the nonzero ratios of dollars to hours are all equivalent to 5.5/1. The ratios 1.5:1 and 5.5/1 are important because they tell us how many of the first quantity for one unit of the second quantity, the **unit rate**. In our examples, the unit rate of cups of flour to cups of sugar is 1.5 and the unit rate of dollars to hours is 5.5. In general, if we know the amount of y corresponding to one unit of x, then we can compute the amount of y for any quantity of x. Therefore, if there are r units of y for every one unit of x, then for m units of x, y will be rm. This number r is the unit rate of y with respect to x.

Unit rates can be computed with either of the two quantities as the basic unit. Using the models in the examples above, we can compute the unit rate of sugar to flour is 2/3 : 1 and hours to dollar is 2/11 : 1. Note that in fractional form, the unit rate of flour to sugar is 3/2 and the unit rate of sugar to flour is 2/3. Similarly, the unit rate of dollars to hours is 11/2 while the unit rate of hours to dollars is 2/11. Because both unit rates are valid, it is very important, in any particular context, to make a choice of the order of the variables and to be clear about which quantity is being the unit. To put it another way, using terminology from 6th grade mathematics: in any context, unless the context dictates which variable is independent, the solver must choose an "independent variable" and stay with that choice throughout the discussion. In these terms the *unit rate* is the quotient r of the dependent variable (y) by the independent variable (x), and is described as r units of y per unit of x.

EXAMPLE 6.

Let's buy some eggs. A measure of the value of a certain quantity of eggs is its number: a dozen eggs has 12 times the value of a single egg, and a gross (twelve dozen) of eggs has 12 times the value of a dozen, or 144 times the value of a single egg. At his roadside stand, the farmer sells his eggs at \$3.84 a dozen. If you want to buy one egg, since it is $\frac{1}{12}$ of a dozen, it will cost you $\frac{1}{12} \times \$3.84 = \$0.32$. If you want to buy a gross of eggs, it will cost you $12 \times \$3.84 = \$46.08$.

In this example we use three different unit rates: cost per egg, cost per dozen and cost per gross, and we have illustrated how to change from one unit to the other. Depending upon the context, the unit measure of eggs could be "one egg" or "one dozen eggs" or "a gross of eggs." If I live alone, the cost per egg is most important to me but if I am a grocer, I want to know the cost per gross. Understanding and clearly communicating the unit rate and which unit it is taken with respect to are vitally important.

EXAMPLE 7.

Camila can ride a bike 5 miles in 20 minutes. Express her average speed as a unit rate of distance over time measured in minutes, and then fill in the missing entries in the table below.

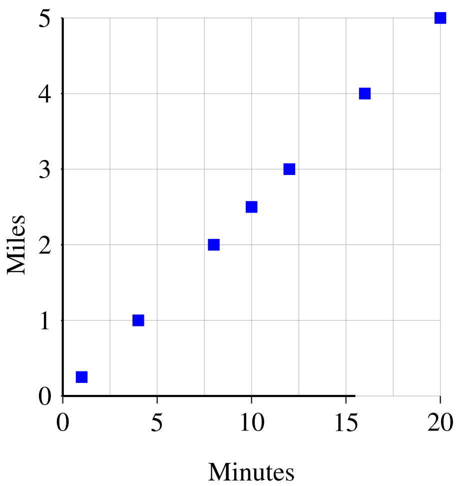
Minutes	20					10	1
Miles	5	1	2	3	4		

Solution. The problem specifies "minutes" to be the independent variable. Since one minute is (1/20)th of twenty minutes, the distance to be associated to one minute is (1/20)th of 5 miles, or a fourth of a mile. So the unit rate is: $\frac{1}{4}$ mile per minute. We now can fill in the table, with the last unit showing unit rate in miles per minute.

Minutes	20	4	8	12	16	10	1
Miles	5	1	2	3	4	2.5	0.25

Again, notice the same patterns in the table as before. As the number of minutes increases by four (the unit rate of minutes per mile), the number of miles increases by one. When the number of minutes triples from four to twelve, the number of miles also triples from one to three. When the number of minutes is halved from 20 to 10, the number of miles is halved from five to two and one-half. The second numeric column shows the unit rate of 4 minutes per mile and the last column shows the unit rate of 0.25 miles per minute. Note the reciprocal relationship of the two unit rates.

When we graph these data, since the original problem was stated in terms of "miles per minute", we take "minutes" as the independent variable and "miles" as the dependent variable.



Miles per Minute

Another use of unit rates is in the analysis of real-world situations. Often we want to find the better deal in the grocery store by comparing two (or more) different sizes or different brands of a given item. Other times we want to determine which object is moving faster or which project is being completed more quickly. When two rates are given, it can be difficult to determine which rate is higher or lower because they have different values. It is not until both rates are converted to the same unit (a unit of one) that the comparison becomes easy. Once we have converted the rates to the same "language," the choice of which size or brand of product gives the best buy is easy. Similarly, once the speeds or rates of completion are measured in the same units, the result is readily apparent. This is illustrated in the next example.

Example 8.

Camila's friend Xavier also likes to ride bikes. He can pedal 4 miles in 15 minutes. Who can ride faster, Camila or Xavier?

Solution. At first glance, it is hard to tell who is quicker. From the given information, Xavier travels a shorter distance, 4 miles instead of 5 miles, but he does so in less time, 15 minutes instead of 20. We recall the computation of Camila's rate (of one mile in four minutes or one quarter miler per minute, that is 0.25 miles/minute). The computation for Xavier is

$$\frac{4\ miles}{15\ minutes} = \left(\frac{4}{15}\right)\left(\frac{miles}{minute}\right) = 0.267\frac{miles}{minute}\ ,$$

so Xavier rides a little faster than Camila (since he covers 0.267 miles in the same time that it takes her to ride 0.25 miles).

Another way to see this is to calculate the time each takes to ride one mile. it takes Camila 4 minutes to ride one mile where it takes Xavier 3.75 minutes to ride one mile. Since it takes her longer, she is slower.

As noted earlier, if we have equivalent ratios, we can use the unit rate to solve for desired quantities. We saw this in the examples involving cookies, yardwork, and cycling. We now introduce the concept of a proportional relationship and take advantage of the unit rate to quickly compute the desired quantity.

Two quantities x and y are in a *proportional relationship* if the quotient y/x is a fixed number r whenever x is not zero. This may also be written $y = rx$ or $x = y/r$ (when r is nonzero). In a proportional relationship, r is the *unit rate* of y with respect to x. This same unit rate, r, is also called the **constant of proportionality** (sometimes referred to as the proportional constant).

Example 9.

Camila's club is sponsoring a "bike-marathon" where cyclists ride for 26.2 miles. Given Camila's average rate of 5 miles in 20 minutes, how much time will it take her to complete the race?

Solution. As noted above, there are two possible unit rates: miles per minute and minutes per mile. Assuming she will be able to cycle at this rate for the entire race, the ratio of time to miles will be a constant, 4 minutes per mile. Letting y be the time, x be the distance and using the constant of proportionality $r = 4$, we quickly compute that $y = 4\frac{min}{mile} \cdot 26.2 mile = 104.8$ minutes or one hour and 44.8 minutes.

Note that the argument says that when we scale up 1 mile to 26.2 miles, the time it takes (4 minutes) scales up to $26.2 \cdot 4$ minutes; in other words, for Camila, "miles cycled" and "time it takes" are both measures of the same distance so are proportional, with unit rate one-fourth mile per minute.

To summarize the content of this section, two variables y and x are said to be *associated,* if the values of these variables are paired. To say that they are in the *ratio* $a : b$ is to say that for every pair of associated values, the fraction y/x is equivalent to the fraction a/b. The *unit rate* of y with respect to x is $r = a/b$. If we shift our interest from specific pairs of values to the behavior of the variables with respect to each other, we shift the language to that of *proportion*. We say that the variables y and x are *proportional* if the quotient y/x is constant for all pairs of associated values. That constant is called the *constant of proportionality*. Thus ratio and proportion describe the relation between two variables that are associated, but put focus on different aspects of that association. .

Section 4.2: Construct and Analyze the Representations of Proportional Relationships

Recognize and represent proportional relationships between quantities. 7.RP.2.

- *a. Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.*
- *b. Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.*
- *c. Represent proportional relationships by equations.*
- *d. Explain what a point (x, y) on the graph of a proportional relationship means in terms of the situation, with special attention to the points $(0, 0)$ and $(1, r)$ where r is the unit rate.*

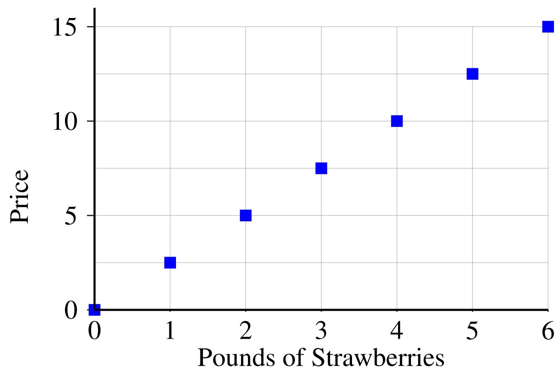
In this section, students build on proportional relationships as developed in section 4.1 by exploring, in contexts, representations of the relationship using tables and graphs. From these tables and graphs, students determine whether a context represents a proportional relationship and begin to create an equation to model proportional situations (further developed in section 4.3). The goal of this section is to have students move fluidly between the table, graph, equation (if proportional) and context.

EXAMPLE 10.

Jonathan loves strawberries. At the store, strawberries sell for \$2.50 per pound. Create a table and then draw a graph for this situation.

SOLUTION.

Pounds of strawberries	0	1	2	3	4	5	6
Price (\$)	0	2.50	5.00	7.50	10.00	12.50	15.00



Cost of Strawberries

Using the data for this problem, we calculate that the ratio of price to pounds of strawberries in each nonzero column is (5/2):1. Because of this constant unit rate, the graph consists of points on a line that goes through the origin, (0, 0). This represents the fact that, if Jonathan doesn't buy any strawberries, he doesn't have to pay anything. Also, note that the unit rate of price to pounds of strawberries is represented at the point (1, 5/2). These two traits, the graph is a line going through the origin and the unit rate at the point (1, r), are characteristics of every proportional relationship.

Note that, when graphing data given in a table, the way the table is set up determines which quantity goes on the horizontal axis and which goes on the vertical. The convention is that the first quantity (on top for a horizontal table and on the left for a vertical table) goes on the horizontal axis. When a problem is given in a context, the solver gets to choose which quantity goes on which axis according to the sense of the context.

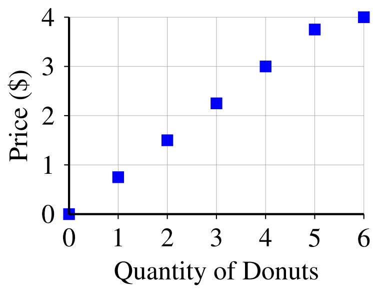
Example 11.

Tasty-Yums sells assorted donuts according to the prices in the table. Determine if price and quantity of donuts are proportional.

Quantity	0	1	2	3	4	5	6
Price ($)	0	.75	1.50	2.25	3.00	3.75	4.00

Solution. Computing the unit rates of price per quantity for 1 to 5 donuts yields \$0.75 per donut. However, the rate for 6 donuts is \$4.00 per half dozen which comes to a per-donut rate of \$0.66. So, the company gives a discount for buying half a dozen donuts, and may advertise this by writing "Donuts: \$0.75 each, or \$4 for a half dozen."

Here is a graph of the data in the above table:



Cost of Tasty-Yums Donuts

Since the last point $(6, 4.00)$ does not fit on the line segment emanating from $(0, 0)$, this is not a proportional relationship once we purchase 6.

The previous two examples show that some relationships are proportional and others are not. Often a proportional relation is not stated, but is implied by giving a few illustrations that show a pattern. Students benefit from determining whether a relationship is proportional by asking: "Is there a constant unit rate?" or "Do the points all lie on a line that goes through (0, 0)?" If the relationship is proportional, we can take the next step and begin to write an equation to express the relationship. This is illustrated in the next example.

Example 12.

Amber is studying the relationship between the distance she covers and the number of steps to span that distance. She assumes that "steps taken" is a reliable measure of distance; that is "steps taken" will be proportional to any measure of the distance covered. This amounts to assuming that each of her steps covers the same distance. The table below shows the information she has gathered from several little walks.

# steps taken	0	10	20	30	40	1		x
# feet walked	0	16	32	48	64		1	

a. Determine the pattern and fill in the rest of the table.

b. Calculate the unit rates from the data.

c. Write an equation showing how one variable is related to the other.

Solution. It is fair to assume that Amber's stride, the distance in feet covered by each step, is constant. Stride, measured in feet per step, is the unit rate we seek. If we take the first (nonzero) data point, we find that feet per step is 1.6, and we note that each given data pair confirms this; our confidence that Amber's stride is constant is rewarded. Notice that the unit rate of 1.6 feet per step is precisely the multiplicative factor to go from the top row of the table to the corresponding cell in the bottom row. Therefore, if x steps are taken, Amber will have walked $1.6x$ feet. This information is included in the table. Note that we are simply scaling up or down the unit rate of 1.6 feet per step taken.

# steps taken	0	10	20	30	40	1	0.625	x
# feet walked	0	16	32	48	64	1.6	1	$1.6x$

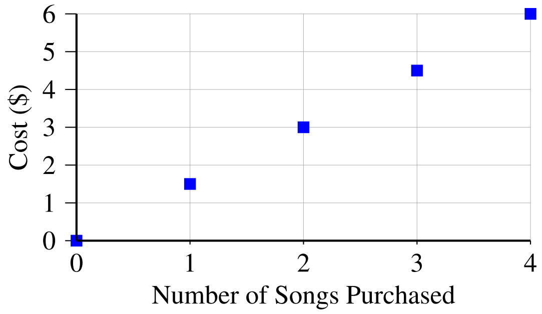
Based on the table, we have found the equation relating number of feet walked to number of steps taken. Letting y be the number of feet walked (second row) and x be the number of steps taken (first row), we see (last data point) that $y = 1.6x$. Note that the unit rate is the coefficient of x (in symbols, $r = y/x$). This will always be the case when the two variables are proportional.

As noted in the introduction, the study of proportional relationships lays a foundation for the study of functions, to begin in Grade 8 and continuing through high school and beyond.

Linear functions are characterized by having a constant rate of change (the change in the outputs is a constant multiple of the change in the corresponding inputs). Proportional relationships are a major type of linear function; they are those linear functions that have a positive rate of change and take 0 to 0.

Example 13.

The graph below shows the cost of the purchase of songs from the online store J-Tunes.



Cost of Purchasing J-Tunes Songs

a. Are the data proportional?

b. Fill in the table below.

# Songs						
Cost ($)						

c. Find the unit rate of cost per song.

d. Create an equation for the cost of purchasing x songs.

Solution.

a. All the data points are on a line emanating from (0, 0) so the number of songs purchased and cost are proportional.

b. Reading from the graph gives the first five entries of the table below.

# Songs	0	1	2	3	4	n
Cost ($)	0	1.5	3	4.5	6	$1.5n$

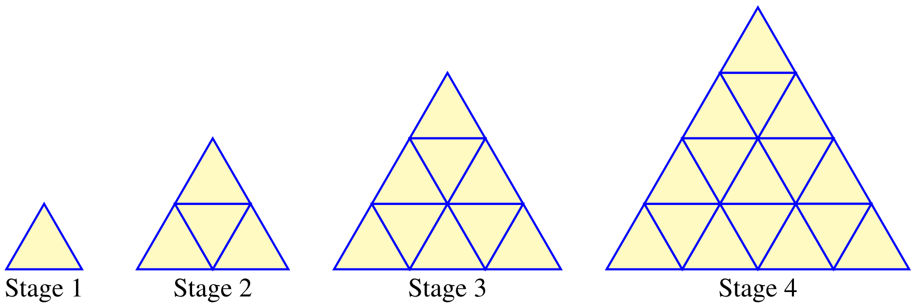
c. The second entry in the table tells us that one song costs $1.50, so the unit rate is 1.5 dollars per song. n songs will cost $\$1.50n$ dollars, as shown in the last entry.

d. Letting C be the cost of the purchase of n songs, we can write this as an equation: $C = 1.5n$.

Now we move to the analysis of scenarios using knowledge about proportional relationships. Geometric configurations provide a good place to explore patterns and reinforce concepts. In the next example, we explore a pattern, look for structure and reinforce the understanding of "perimeter."

Example 14.

Jennifer's young cousin is playing with triangular blocks. She makes the following figures and asks Jennifer if there are any patterns.



a. Fill in the table relating stage to perimeter and number of triangular pieces.

Stage	1	2	3	4	5	6	7	8
Perimeter	3	6						
# pieces	1	4						

b. Create graphs relating stage to perimeter and stage to number of triangular pieces.

c. Make an educated guess (using the above figure) that relates perimeter to the stage number.

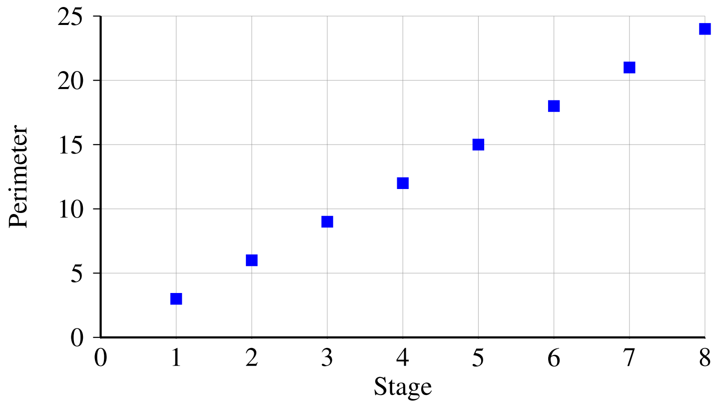
d. Are there any proportional relations in these data? Explain why or why not.

Solution. Notice that units have not been specified, because we are interested in how the figures are numerically related, and whether the measure of the sides is in feet, yards or kilometers is not of interest. So we implicitly take, as the unit of length, the length of a side of the triangle in Stage 1.

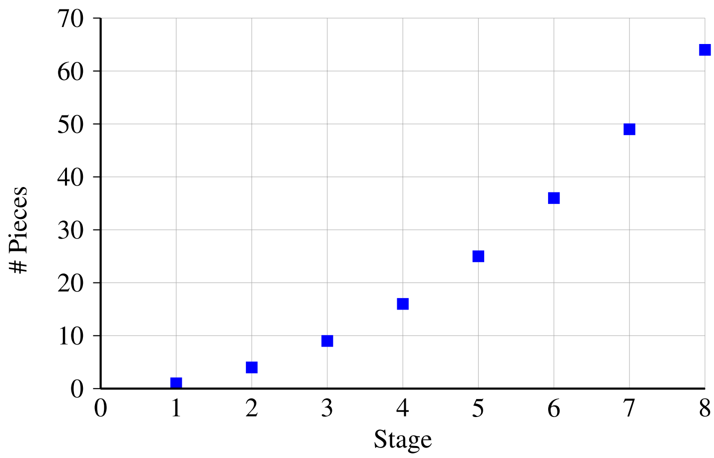
a. The first four data points are found by counting. We can continue by repeating the pattern four more times, or by noting that we move from one stage to the next by drawing a new row of triangles at the bottom. One way of describing the transition from one picture to the next is that it consists of adding a new (bottom) row. The new row consists of flips of all the triangles in the previous bottom row plus two new triangles at the ends. So, the number of triangles on the new bottom row is the next odd number. That tells us how to fill in the bottom row: calculate each entry, by adding the next odd number to the entry just calculated. As for the perimeter, we notice that the move from one stage to the next adds one Stage 1 side length to each side. Since there are three sides to the large triangle, the perimeter increases (from one stage to the next) by 3 Stage 1 side lengths. Thus we finish the second row of the table by adding 3 as we move from Stage to Stage.

Stage	1	2	3	4	5	6	7	8
Perimeter	3	6	9	12	15	18	21	24
# pieces	1	4	9	16	25	36	49	64

b. Graph



Stage and Perimeter



Stage and # Pieces

c. As we look at the table, for every column the entry in the second row (perimeter) is 3 times the stage number. That tells us that $P = 3S$, and so P and S are in the same rato, 3:1. As for the number of pieces per stage, the ratio changes from column to column, so the number of pieces is *not* proportional. We should recognize the pattern of the squares of the stage number.

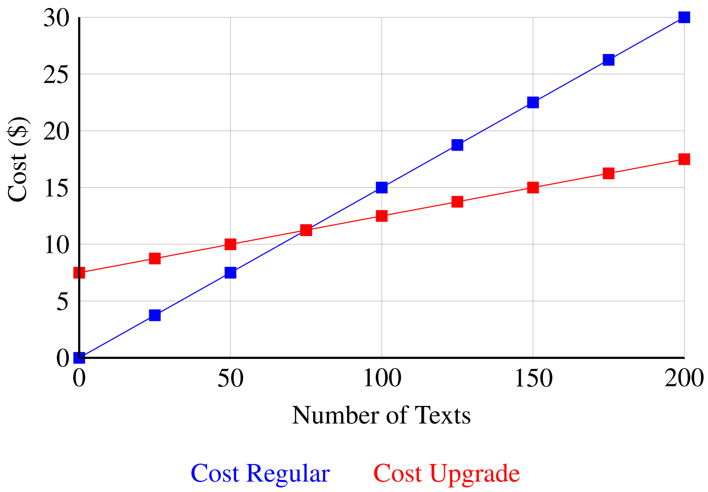
d. The graphs of the data indicate to us that perimeter is proportional to stage number (with ratio $3 : 1$) while number of pieces is not proportional to stage number.

One of the practical uses of finding equations is the comparison of data. Real world examples include comparing different financial options. Below is such an example: comparing cell phone plans. Note that the graphs of the two options are both straight lines, but only one is the graph of a proportional relation.

Example 15.

Grace is researching texting options for her cell phone. With her carrier she pays \$0.15 per text on the basic plan. She can upgrade to a texting plan by paying a flat \$7.50 each month and just 5 cents for each text sent. Create a table to analyze the costs of texting on each plan. Which plan should Grace sign up for? Your answer will reference how many texts she plans to send/receive each month.

Solution. The graph that shows these data together is this; looking at the graph we note that Grace is spending less on the regular plan so long as she sends no more than 75 texts a month. If she sends more than that, the regular plan is more costly.



Section 4.3: Analyze and Use Proportional Relationships and Models to Solve Real-World Problems

Use proportional relationships to solve multistep ratio and percent problems. Examples: simple interest, tax, markups and markdowns, gratuities and commissions, fees, percent increase and decrease, percent error. 7.RP.3

The concepts studied in the previous sections will be applied throughout this section. Students will set up and solve proportions for real-world problems, including problems with percentages of increase and decrease. It's likely, especially for advanced classes, that before the topic is formally introduced students will set up proportions and solve them (a) using properties of equality, (b) by finding a common denominator on both sides, or even (c) by cross multiplying, if they have seen this in previous years or at home. Any method that can be justified using proportional reasoning can lead to a meaningful discussion. Students should be encouraged to justify their answer with other representations (graph, bar model, table, or unit rates) . Example 16.

During the summer, Jordan helps a neighbor with yard work. One day Jordan worked for three hours and earned \$16.50. At this rate, how long will Jordan need to work to have enough money to buy an MP3 player that costs \$88? Set up a proportion equation for this situation and solve.

Solution. First we calculate, using the given data, the unit rate of money earned per hour worked:

$$\text{unit rate} = \frac{\$16.50}{3\ hours} = \$5.50 \text{ per hour.}$$

So, if Jordan works two hours, he earns $2 \times \$5.50 = \$11$; if he works 6 hours, he earns $6 \times \$5.50$, and if he works x hours, he earns $x \times \$5.50$. Since he wants to earn \$88, we should solve $5.50x = 88$, giving $x = 16$ hours.

Example 17.

In a classroom of twenty-seven students, the ratio of those who do not own a cell phone to those who do is 1 : 2. How many students in this class own a cell phone?

Solution. First note that the ratio 1 : 2 is that of the number of students without cellphones to the number of students with cellphones. Let x denote the number of students without cellphones. Then there are $2x$ students with cellphones and a total of $x + 2x$ students. Since that number is 27, we must solve $x + 2x = 27$, giving $x = 9$. We conclude that there are 9 students without cellphones and 18 with cellphones.

Ratios often occur in this way to describe the proportion of people in a given population that have a certain attribute as opposed to those who do not. For example, the ratio of right-handed people to left-handed people is 9 : 1. This is called a *part-to-part* ratio, in that it compares the sizes of different parts of a given population. If instead we said, "one in ten people is left-handed," we'd be saying the same thing, but as a *part-to-whole ratio*. In both cases, we are saying that 90% of the population is right-handed. Be careful in working with ratios expressed in this way. The statements, "there are 9 times as many right-handed people as left-handed" and "10% of the population is left-handed" seem not to say the same thing, but they do.

Example 18.

The weather data for July in Provo, Utah show that for every 7 mostly clear days there are 3 overcast days. This datum is expressed as a part-to-part ratio. To rephrase as a part-to-whole ratio, we might say that we can have a picnic in Provo in July, 7 out of 10 days. Another way: typically in Provo, July will have about 9 overcast days.

Proportional relationships come up when considering tax and commission rates, markups and discounts. In all of these cases, the information is given as a unit rate expressed as a percentage. That means that the "unit" is taken to be 100; a tax rate of 6.75 percent just says that the tax due on \$100 is \$6.75. We have seen a set of problems about percentages in chapters 1 and 3, here we point out that in those problems we are discussing proportional relationships without mentioning proportion. Let us now look at some of these examples of proportional reasoning.

Example 19.

Markup in the retail industry means the percentage of cost for an item that is added to set the sale price of the item. To illustrate, suppose the markup is 12 %. That tells us that for every \$100 spent on acquiring goods, the manager adds on \$12 to create the store's profit. That is the same as saying that the markup is "12 cents on the dollar" – meaning that for an item that costs the store, say \$88, the markup will be $0.12 \times 88 = 10.56$ dollars.

In general, if the markup is $m\%$, then the sale price for an item costing C dollars will be

$$C + \frac{m}{100}C$$

EXAMPLE 20.

A sporting goods store is advertising a sale of 30% off every regularly priced item. Kelly wants to purchase some new apparel and equipment.

a. Use a proportion to solve for the sale price of a shirt that is normally priced at \$19.50.

b. Complete the following table using your choice of solution method.

Original Price (x)	\$19.50	\$25.00	\$43.20	\$9.90	\$8.48	\$29.99
Sale Price (y)						

c. What is the unit rate of (sale price)/(original price)?

d. Write an equation for y, the sale price, for an item that has a regular price of x.

e. If the sale price for running shoes is \$48.23 what was the original price?

SOLUTION.

a. If the regular price is \$19.50, then the discount, d, (the part) can be found using the percent proportion equation

$$\frac{30}{100} = \frac{d}{\$19.50} \quad \text{so} \quad d = 0.30 \cdot 19.50\ .$$

Solving gives $d = \$5.85$. Since the discount is \$5.85, the sale price must be \$19.50 − \$5.85 or \$13.65. Alternatively, one might look at the sale price, s, as the part. Since there is a 30% discount, Kelly will pay 100% − 30% = 70% *of the original price*. The percent proportion equation would be

$$\frac{70}{100} = \frac{s}{\$19.50}$$

Solving this proportion yields $s = \$13.65$, the same result.

b.

Original Price (x)	\$19.50	\$25.00	\$43.20	\$9.90	\$8.48	\$29.99
Sale Price (y)	\$13.65	\$17.50	\$30.24	\$6.93	\$5.93	\$20.99

c. Since we know this is a proportional relationship, it suffices to calculate the ratio for one instance, the unit rate is $13.65/19.50 = .70$, or 70%. Wait a minute – was there an easier way to compute this?

d. Letting S represent the sale price and R the regular price, $S = 0.7P$.

e. In this part of the problem we are given the sale price, S of \$48.23, and are looking for the original price P. So, we have to solve $48.23 = 0.7P$. One way to figure this out is to rearrange the previous equation using the multiplicative inverse of 0.7 to get $P = \frac{10}{7}S$. Then we substitute in 48.23 for S

and get $P = 68.90$. We can check our answer by multiplying 68.90 by 0.7 to get 48.23. Therefore we have verified that the original price was \$68.90.

In general, if the discount on an item is $d\%$, then the sale price for an item originally priced at S dollars will be

$$S - \frac{d}{100}S$$

Example 21.

If the population today of Isla Incognita is 113% of what it was in the year 2000 and the population in 2000 was 36,000, then today's population is $1.13 \times 36,000 = 40,680$. Suppose I know the present population of Isla Linda to be 18,000, and I also know that it is 113% of the population in 2000. What was the population in 2000?

Solution. We have to solve the equation $1.13 \times P = 18,000$, or $P = 18,000/1.13 = 15,929$. Be careful! You might have argued, "if the population grew by 13% from 2000 to today, that means that the 2000 population was 13% less then," so you calculate $(1 - 0.13) \times 18,000 = 15,660$, a quite different figure, and the wrong one.

This possible mistake is best illustrated by a simple example. If there are three people in a room, and another person enters the room, the population increases by 33%, for 4 is $4/3 = 1 + 1/3$ of 3. But now, if that person leaves the room, the population decreases by 25% because $3 = 3/4$ of 4, or $1 - 1/4$ of 4.

Example 22.

The population of the US is 313.9 million, and that of Brazil is 190.7 million. Express in percentages how much larger the US is than Brazil and how much smaller Brazil is than the US.

Solution. Choose p so that the US is $p\%$ larger than Brazil. Then we have to solve

$$\left(1 + \frac{p}{100}\right)(190.7) = 313.9$$

Now let q be such that Brazil is $q\%$ smaller than the US. Here we have to solve

$$\left(1 - \frac{q}{100}\right)(313.9) = 190.7$$

The answers are $p = 64.6$ and $q = 39.2$.

Example 23.

Two stores have the same skateboard on sale. The original price of the skateboard is \$200. At store AAA, it is on sale for 20% off with a rewards coupon that allows the purchaser to take an additional 30% off the sale price at the time of purchase. At store BBB, the skateboard is on sale for 30% off. They too have a rewards coupon, but their coupon is for an additional 20% off the sale price. Will the price for the skateboard be the same at both stores? If not, which store has the better deal?

Solution. The problem presents two scenarios, at stores AAA and BBB, and asks us to compare the final price in each scenario. Let us go through each scenario in detail.

- At AAA, the original price is \$200. It is marked down 20%, which is $0.20 \times 200 = 40$ dollars, so is now priced at \$160. The purchaser with coupon can take off an additional 30%, which is $0.30 \times 160 = 48$ dollars. Thus the final price to the purchaser with coupon is \$112.

- The scenario at BBB is the same, but with the percentages interchanged. So, at BBB the first markdown is $0.30 \times 200 = 60$ dollars, and the sale price is \$140. The purchaser with coupon gets a further 20% discount, which is $0.20 \times 140 = 28$ dollars. Thus the final price is \$112.
- The result is that the deals are the same. There is a way to see that more easily. At AAA, the sale price is 0.80×200, and with the coupon, the final price is thus $0.70 \times 0.80 \times 200$ dollars. At BBB, the percentages are reverses, so the final price is $0.80 \times 0.70 \times 200$. But these numbers are the same because of the commutativity of multiplication.

Chapter 5
Geometric Figures and Scale Drawings

Here we connect concepts developed about ratio and proportion in the previous four chapters to concepts in geometry. In the first section we start by exploring conditions necessary, in both angle measure and side length, to construct unique triangles with ruler and protractor. The concept of 'uniqueness' is discussed as an introduction to the idea of equivalence under a rigid motion. Students will distinguish with more precision than in previous years that two figures can be exactly the same size and shape, or can be the same shape, but different size, or can be of different shape. The focus of the second section is on polygons that are the same shape but different size. Students construct scaled drawings of triangles first and then other figures and through explorations note that objects that are the same shape but different size have angle measures that are the same and side lengths that are proportional. Essential here is the notion of scale. Students will connect ideas of scale to ideas associated with ratio and proportion to reproduce images noting that side lengths change by the same factor but area changes by the square of the factor. In the third section, we turn to circles and observe that all circles are scaled drawings of each other; from which it follows that for any circle, the circumference (length of the perimeter) is proportional to the length of the radius, and the area is proportional to the square of the radius. Students will discover the remarkable fact that the constant of proportionality for the circumference is twice that of the area. In fact, we have $A = \pi r^2$ and $C = 2\pi r$, where π is the area (in square units) of a circle of radius 1 unit. The chapter ends with students examining angle relations as a means to solve problems, a theme to be further explored in the next chapter.

Students will also observe that there are many triangles with given angle measures at the vertices, and that they are all scale drawings of one another. This is a significant characteristic of similarity that shall be further explored in grades 8 and 9; in grade 7 we simply observe that it is true for the triangles that we construct with ruler and protractor. In section 4, we gather together, through exploration, other statements that appear to be true: for example that the sum of the angles of a triangle is a straight angle, and use that fact to solve problems involving angles.

In elementary school students have found the area of rectangles and triangles. They have measured and classified angles, and drawn angles with a given measure. They have learned about circles informally, but haven not learned rigorous definitions of Circumference and Area.

The term "similarity" is defined in the Grade 8, using the concept of "dilation" along with the rigid motions. In grade 7 we use terms such as "same shape" and "same size." Grade 7 students will observe that the sum of the angles in a triangle is 190^{circ}, and in Grade 8 students will justify this and extend that knowledge to exterior angles and interior angles of other polygons. Additionally, Grade 8 students will extend their understanding of circles to surface area and volumes of 3-D figures with circular faces. Grade 9 students will formalize the triangle congruence theorems (SSS, SAS, AAS, ASA) and use them to prove facts about other polygons. Also, Grade 8 students will e extend the idea of scaling to that of dilation of right triangles and then to the slopes of lines. Grade 10 students will formalize dilation with a given scale factor from a given point as a non-rigid transformation (this will be when the term "similarity" will be defined) and will solve problems with similar figures. The understanding of how the parts of triangles come together to form its shape will be deepened in Grade 8 when students learn the Pythagorean Theorem, through to Grade 11 and trigonometry (numerical geometry of the right triangle) and its generalization to all triangles through when they learn the Law of Sines and Law of Cosines.

Geometry is the study of shapes and forms with attention to defining properties and relationships among them. In the elementary grades, student have learned much about these forms and their properties, in terms of lengths, angles and area. In this chapter, and again in the geometry chapters of 8th grade, we undertake a review of this knowledge, and start to give it some logical structure that finally will be fully studied in secondary mathematics. Here we will rely on constructions and diagrams to illustrate and explore concepts. While emphasizing that all geometric knowledge comes out of understanding these constructions, we must caution that a good picture is just an example, and each picture will have features that are not characteristic of the situation prescribed by the context. Nonetheless, working with diagrams is an essential component of geometric thinking.

Geometry of the plane was well understood in antiquity. When Alexander the Great, toward the end of the 4th century BCE, founded the library at Alexandria the Greek philosophers and mathematicians moved there to set up their schools. They set as a primary goal the creation of an exposition of plane geometry in the strict logical style advocated by Aristotle. This was the "Elements" of Euclid, which remained the standard exposition until today. At the beginning of the 20th century CE, David Hilbert wrote what was to become the definitive Euclidean geometry in this logical format. Around the same time, the mathematician Felix Klein suggested a new way of looking at geometry – as the study of objects in a set that are unchanged by a particular collection of transformations of the set. According to Klein, the fundamental objects in the study of planar geometry are not the axioms and theorems, but the rigid motions: rotations, shifts and reflections. Two objects are considered *congruent*, of the same shape and dimension, if there is a rigid motion taking one onto the other. Similarly, the fundamental objects in spherical geometry are the rotations of the sphere, and so forth. This perception of geometry is most useful in its applications, and, in particular, forms the basis for the online applications for geometry (Geogebra, Geometer's Sketchpad, etc.). For that reason, as well as the closer correlation to intuition of the axiomatic approach, transformational geometry has been adopted by the Common Core, and the Utah Core Standards as the basis for the exposition of geometry starting in seventh grade and going through secondary mathematics.

Section 5.1. Constructing Triangles from Given Conditions

In this section students discover the conditions that must be met to construct a triangle, first using only straightedge and compass, and then introducing measure through ruler and protractor. It is important to keep in mind the difference between the "thing" and the "measure of the thing," in particular, when discussing line segments and angles. A line segments has a certain measure, its length, and an angle has its measure (degrees, and much later, radians). This numerical quantification of geometric concepts is relatively new in human history, relative to the understanding of the basic facts relating lines and angles. For a carpenter, a plank is of a certain length, width and thickness, but also of a certain cost and a certain material. These measures of a plank are its characteristics, and distinct from the object. If the carpenter says that "here we will use 137 linear feet of plank," that gives us some information, but not the information about material, the strength of the material, and its cost. Making this point here helps immeasurably later.

By constructing triangles students will note that the sum of the two shorter lengths of a triangle must always be greater than the longest side of the triangle and that the sum of the angles of a triangle is always a straight angle($180°$. They then explore the conditions for creating a unique triangle: three side lengths, two sides lengths and the included angle, and two angles and a side length, whether or not the side is included. This approach of explore, draw conclusions, and then seek the logical structure of those conclusions is integral to the new core. It is also the way science is done. In later grades students will more formally understand concepts developed here.

Throughout this chapter, students and teachers will use geometric terms with which they have become familiar: point, line, line segment, circle, etc. Though in 7th grade these terms will not be rigorously defined, it is important that they are used correctly and misconceptions are not developed, thus we take time here to provide a frame for using terms.

The most fundamental objects in the geometry of the plane are points, lines and circles. It is important to distinguish between a drawing of a point and a mathematical point, in that geometric points are ideal and have no size while the drawings we make do. In the same way, a drawn line segment will have thickness, but the ideal concept does not.

A *line segment* is determined by two points, called its *endpoints*. The line segment between two points is drawn with a straight edge aligned against the two points, and its length is measured by a ruler.

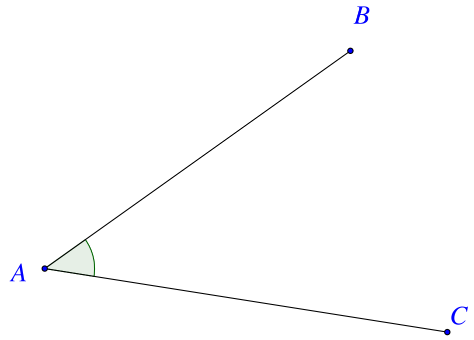
A *circle* s defined as all points of equal distance to a **center** point. A circle is drawn with a compass: the needle-point is situated at the *center* of the circle, and the pencil point traces out a curve as it is rotated around the fixed center. When we speak of "the area of a circle," we are referring to the area of the region enclosed by the circle. Any line segment from the center of the circle to that curve is a *radius*; all radii have the same measure, denoted by r. The curve that bounds the circle is called its *circumference*.

Once unit lengths have been chosen, distance on the plane is measured using a ruler whose markings are based on the chosen unit. Thus, we might have a yard ruler or a meter stick; in either case it is important to understand that it is the distance between two points (or the length of the line segment) that is being measured, and (as pointed out in chapter 4), any two ways of measuring distance are proportional. When it comes to curved lines,like circles, there is no easy, ruler-like way to measure their length. We will discuss this further for the circle in the third section.

A *ray* is a piece of a line that extends from one point (called the *vertex*) on and on in only one direction. We name rays by listing the initial point or vertex first, so ray AB has vertex A and extends on in one direction through the point B.

Angles

An *angle* consists of two rays which share the same vertex. The rays are called the sides of the angle. The angle with rays AB and AC is shown in figure 1. We refer to this angle using the symbol $\angle$, as $\angle CAB$. Note that when we name an angle, the vertex is listed in the middle, and the other outside letters designate points on the defining rays. So, the symbols $\angle CAB$ and $\angle BAC$ denote the same angle; in other words, we do not distinguish the way the angle is traversed (clockwise or counterclockwise). The distinction will become important in 8th grade when we discuss orientation,



We measure angles in degrees using a protractor. A full circle rotation around a point is assigned the measure of 360°. The reason for this is historical and dates back to the times of the ancient Babylonians. If the rays of an angle lie on the same line, but point in opposite directions, the angle is half the full rotation, and so has 180° and is called a *straight angle*. If two lines intersect at a point and the angles at the point of intersection created by those lines are all equal, then they all have measure of 90° is known as a *right angle*; these are called *right* angles.

We classify angles in reference to these designations. An *acute angle* measures less than 90°. An *obtuse angle* measures greater than 90° and less than 180°.

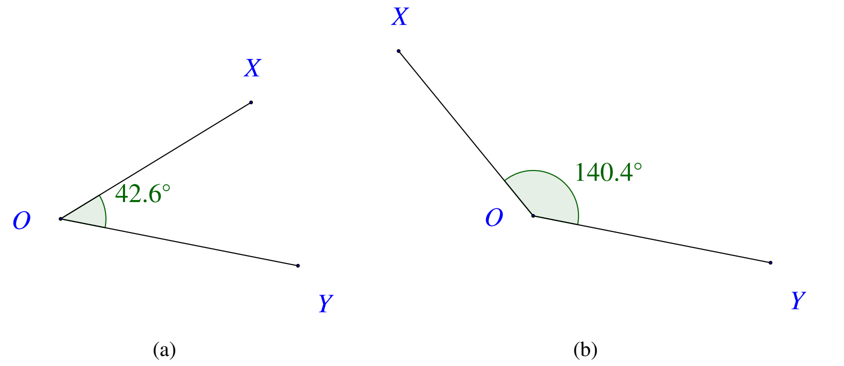


Figure 2: (a) an acute angle, (b) an obtuse angle

(The example angles shown here were constructed and measured using Geogebra.)

It is important that students to acquire facility in using tools and technology in mathematics, especially for the ability to draw geometric figures, to illustrate concepts and to solve problems. The classical tools of plane geometry are the straightedge and compass; the tools for measurement are ruler (distances) and protractor (angles). It is important to learn how to use these tools, even though these tasks are greatly simplified through modern technology. For that reason it is important to become acquainted to the many only programs for drawing and analyzing geometric constructions; to name a few: Excel, Geogebra, Geometer's Sketchpad, Maple and Mathematica. Excel is noteworthy in the sense that the software is at a basic level, and so a lot of the work of creation of a good image is left to the student. Geogebra and Geometer's Sketchpad are very sophisticated instruments, allowing for dynamic manipulation of drawings; as such they can provide real insight into the concepts and procedures of geometry. Maple and Mathematica are research-level tools, incorporating all kinds of graphing capability, but also great facility in numeric and symbolic computation. In an appendix, we have provided a basic introduction to the use of hand-held tools as well as Geogebra.

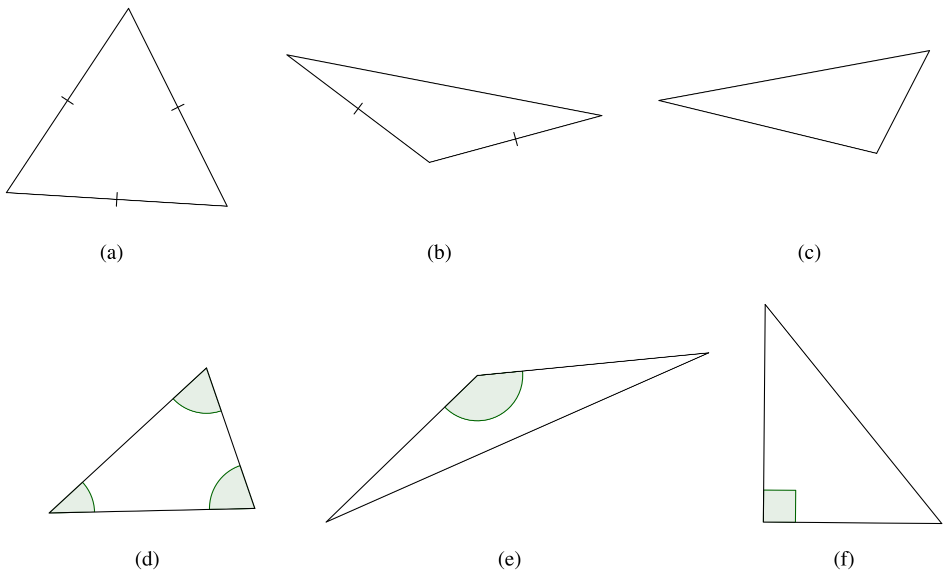


Figure 3: Triangles

Triangles

A *triangle* is a region in the plane enclosed by three line segments. Figure 3 (on the preceding page) illustrates several types of triangles. When two or more sides have a hash mark, that means that those line segments are of equal length.

Triangle (a) is equilateral, (b) is isosceles, (c) is neither (scalene), (d) is acute (all angles are acute), (e) is obtuse (one angle is obtuse), and (f) is a right triangle (the angle marked is the right angle).

Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle.7.G.2.

Let us pause to introduce (or remember) certain vocabulary which will make it easier to talk about triangles. A *vertex* of a triangle is a point where two sides meet. A triangle has three vertices and three sides. Typically, the vertices of a triangle are labeled with capital letters, such as A, B, C, and the *opposite side* by the corresponding lower case letter (a, b, c. Given two vertices, the *included side* is the side joining the two vertices, which is also the side opposite the third vertex. Given a side, the *adjacent vertices* are the vertices at the ends of the side. Finally, we use the symbol Δ to designate a triangle; so ΔABC means the triangle with vertices A, B, C.

The question that students will now explore is this: given three positive numbers, a, b, c, is there a triangle with sides of these lengths? First lets look at the case where the lengths are the same.

Example 1.

Given a length a, how many triangles are there with all sides of length a?

An important question here is: what do we mean by "how many?" For example, triangle (a) above is a triangle all of whose sides are of the same length. If we move triangle (a) horizontally, do we get a different triangle? If we move triangle (a) vertically, or in any direction, should we call that a different triangle? We'd rather not: we want to say that these are the same triangles, only in different positions. Similarly, if we rotate the triangle around some point, once again we get the same triangle, but in a different position. So, let's rephrase our question:

Let $a > 0$ be a positive number. On a piece of graph paper, let A be the point on the horizontal axis of distance a from the origin O. How many triangles are there with one side OA, and all sides of the same length?

Solution. Draw, with a compass, or with appropriate technology, the circles of radius a centered at O and A. These circles will intersect at two points; one above the horizontal axis, and one below. Call these points B^+ and B^-. These are the only possibilities for the third vertex of the triangle (see Figure 4).

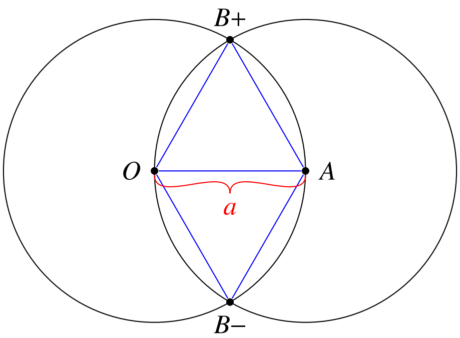


Figure 4

Are these triangles "different?" Not really, because one is the reflection of the other in the horizontal axis . So, we can conclude:

> Given a length $a > 0$, we can construct a triangle of side length a with one side on the horizontal axis with the origin as one endpoint, so that every triangle of all side lengths equal to a can be moved by rotations and slides to this one, or its reflection in the horizontal axis.

We now turn to consider general triangles, exploring what conditions suffice to construct a triangle, and in what sense it is unique (the only solution possible, ignoring its position on the plane). First, we ask if there are conditions on a set of three positive numbers for them to be the lengths of the sides of a triangle. Try the lengths 3, 6, and 10 units, and then lengths 3, 9, 10 units. We see inFigure 5, that we cannot find a triangle with sides given by the first set of numbers, but we can for the second.

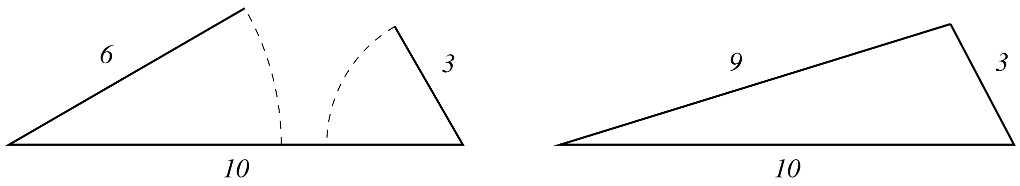


Figure 5

Example 2.

What are the possible values for the third side of a triangle if the other two sides are 2 and 12?

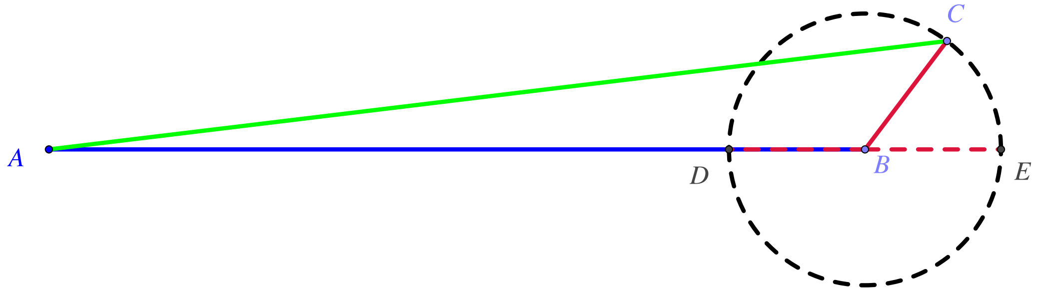


Figure 6: Building a triangle with side lengths 12 and 2.

In Figure 6, AB is the side of length 12, and BC the side of length 2. Imagine swinging the segment BC around point B, then point C will always be somewhere on the circle shown. No matter what angle we choose between the two segments, the third side of the triangle must connect point C to point A. For all points C on the circle (except D and E) there is a triangle with side lengts 2 and 12. The triangle (except for the possibility of flipping in the line $ADBE$) is unique. Now, the shortest line segment between A and a point on this circle is AD of length 10 units, and the longest such line segment is AE of length 14 units. Since the third vertex of our triangle cannot be either D or E (for in those cases all sides of the triangle lie on the same line), we can conclude that the third length must be strictly between $10 = 12 - 2$ and $14 = 12 + 2$.

There was nothing special about the numbers 12 and 2 in this argument, we can replace them with any two positive numbers a and b with $a \geq b$, and assert if c is the length of the third side of a triangle with sides of length a and b, we must have $c > a - b$ and $c < a + b$. An easier way of stating this is:

> For any triangle, the sum of the lengths of two sides is greater than the length of the third.

This is called the *triangle inequality*. Another way of saying this is that the longest side length of a triangle is less than the sum of the lengths of the other two sides. This observation can be extended to arbitrary polygonal paths, showing that the total length of such a path is no less than the length of the straight line between its endpoints. Consequently the length of any polygon side is always less than the sum of the other polygon side lengths.

Now, through exploration, students will make this important observation:

> If a, b, c are three positive numbers satisfying the triangle inequality, then there is a unique triangle (up to motions in the plane) with those numbers as side lengths.

To see this, pick three numbers a, b, c that satisfy the triangle inequality. On a coordinate plane, label the origin as A and label a point B on the positive horizontal axis so that the line segment AB has length a (Consult the above figure, but with 12 replaced by a and 2 replaced by b) . Now, draw a circle with center at B and of radius b. Because of the triangle inequality, c is between $a - b$ and $a + b$, so there is a point C on the circle above the horizontal axis that is of distance c from A. These three points are the vertices of a triangle of side lengths a, b, c. Now, suppose that we have another triangle with these side lengths. We can move (by a slide and rotation) that triangle so that the side of length a coincides with the segment AB. Then the side of length b has an endpoint at A or B. If it is at A, reflect the triangle in the horizontal line through the midpoint of AB. Now, the side of length b has B as an endpoint, and the side of length c has one endpoint at A and the other on the circle of radius b centered at B. But there is only one point on that circle whose distance from A is c, so the moved triangle coincides with the triangle we constructed.

Oops, not exactly - there is a point C' on the circle lying below the axis of distance c from A, that forms the triangle $\Delta ABC'$. But, this is the reflection of ΔABC in the horizontal axis, so is still the same triangle as constructed.

Here is another set of conditions for which there is a unique triangle satisfying the conditions:

> Given an angle $\angle ABC$, and positive numbers a and c, then there is a unique triangle ΔABC with the given angle , and the sides adjacent to that angle of lengths a and c.

Measure off a distance a on the ray BA from the point B, and measure off a distance c on the ray BC from the point B. Draw the line segment joining the endpoints of those segments to get the desired triangle (see Figure 7). Now, this is the unique triangle satisfying the given conditions, because if we have another such triangle we can move the angle to the angle ABC, and the side of length a is either on the ray BA or the ray BC. If it is on the first ray the two triangles coincide. But what if the side of length a is on BC, do we get a “different” triangle?

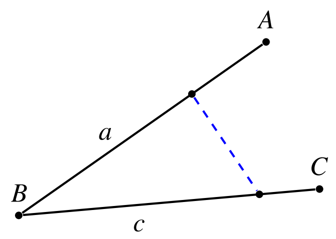


Figure 7

If we are given two side lengths and an an angle, in order that they describe a unique triangle it is important that the lengths be of the adjacent sides, as we see in Figure 8: there is one acute triangle and one obtuse triangle with given angle and side lengths.

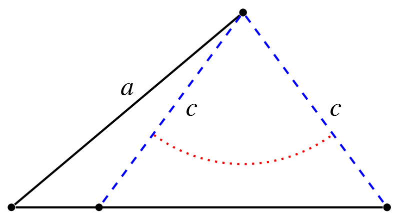


Figure 8

Example 3.

> Given two angles, and a positive number a,if there is a triangle with a side of length a whose adjacent antes are the given angles , then it is unique.
>
> Solution. Draw a line segment AB of length a, and copy the angles (as shown in Figure 9) at the endpoints of AB.

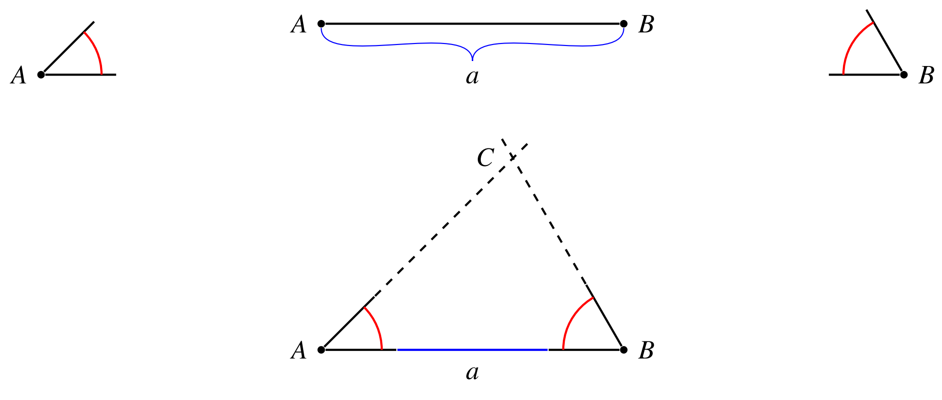


Figure 9

> If the rays of the angles, other than the ray containing AB intersect, call the intersection point C, then ABC is a triangle.

This explanation is too easy, with some gaps in the logic. This might be a good time to explore them in a preliminary way. First of all, figure 9 is not the only possibility: if the two given angles are obtuse, then there will be a point of intersection on the side of AB opposite the one depicted. In this case, the given angles are *exterior* angles of the triangle. A student may observe that if the given angles are right angles, there will be no point of intersection C. This generalizes to: if the rays in question are parallel, there is no point of intersection and thus no triangle. It may come up that this statement is the same as saying that the sum of the whose measures of the given angles is 180°. The final realization that might happen is that the condition for there to be a point of intersection is that the sum of those measures is less than 180°, In any case, these two examples should help the students understand that if the measure of two angles of a triangle is known, then so is the third.

Students will observe that the the condition on the measures of the angles leads to an intersection point of the two outside rays of the angles. Denoting that point as C, the triangle ΔABC is the desired triangle.

At this point, the student may wonder: why did this turn out this way? That is a question that will be answered in secondary mathematics; for the time being, what is important is that students explore these ideas through constructions of their own. A partial answer to that question will come later in grade 7: that the sum of the measures of the angles of a triangle is 180°, so the sum of the measures of two angles has to be less than 180°. In fact, students will see that it follows from this fact, that if we know the length of one side and the measures of *any* two angles , then the triangle with these dimensions is unique.

The following animations at the website *Mathsonline* show how to construct triangles when certain measures are given. Before looking at the links, try the following using a ruler and compass. You may use a ruler, a protractor, and a compass.

- Construct a triangle with sides of length 10 cm, 12 cm, and 18 cm.
  Side-side-side animation here:
  `http://www.mathsonline.org/pages/animationPage.html?triangle3sides`
- Construct a triangle with 2 sides of length 15 cm and 9 cm and an included angle of 35°.
  Side-angle-side animation here:
  `http://www.mathsonline.org/pages/animationPage.html?triangle2sides`
- Construct a triangle with a 25° angle, a 10 cm included side, and a 100° angle.
  Angle-side-angle animation here:
  `http://www.mathsonline.org/pages/animationPage.html?triangle1side`

Here are some more facts about constructing figures that students might come across in their explorations;:

- It is not true that there is a unique triangle with given angles, as Figure 10 shows. The question: "what do such triangles have in common?" will be taken up in the next section, and discussed in detail in 8th grade.

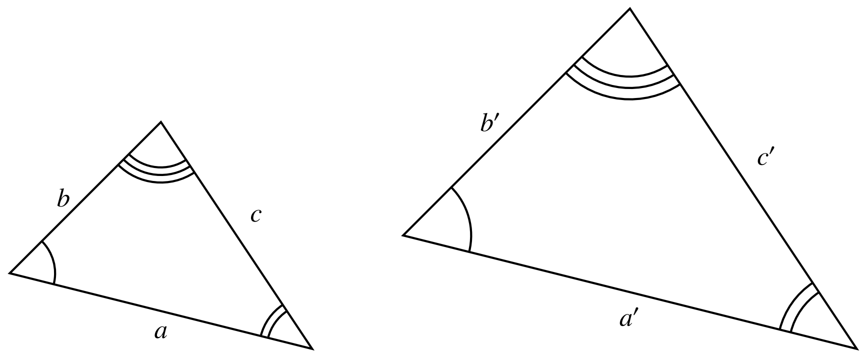


Figure 10

- There is a unique circle with given center and radius. Given two circles of the same radius, we can slide the center of one to coincide with the center of the other. Then the circles coincide as well.
- Given two positive numbers a, b we can construct a rectangle with side lengths a and b: just go from the origin along the horizontal axis a distance a, and along the vertical axis a distance b. These are two sides of the desired rectangle. Furthermore, given any other rectangle of side lengths a and b, we can move it by a slide and a rotation so that it coincides with the constructed rectangle.

Section 5.2. Scale Drawings (Objects that have the same shape)

The central idea of this section is scale and its relationship to ratio and proportion. Students will use ideas about ratio, proportion, and scale to a) change the size of an image and b) determine if two images are scaled versions of each other.

By the end of this section, student should understand that we can change the scale of an object to suit our needs. For example, we can make a map where 1 inch equals one mile; lay out a floor plan where 2 feet equals 1.4 cm from a diagram; or draw a large version of an ant where 3 centimeters equals 1 mm. In each of these situations, the "shape characteristics" of the object remain the same, what has changed is size. Objects can be scaled up or scale down. Through explorations of scaling exercises, students will see that all lengths of the given object are changed by the same factor in the scaled representation; that the factor is called the *scale factor*.

The term "similar" is not defined in Grade 7; here students continue to develop an intuitive understanding of "the same shape," so that the concept of similarity (introduced in Grade 8) will come naturally. Throughout this section students should clearly distinguish between two objects that are of the same shape and dimension and objects that are scaled versions of each other. In particular students will come to understand that two polygonal figures that are scaled versions of each other have equal angles and corresponding sides in a ratio of a:b where $a \neq b$ Students should also distinguish between saying the ratio of object A to object B is a:b and the scale factor from A to B is b/a. This idea links ratio and proportional thinking to scaling.

Students will learn to find the scale factor from one object to the other from diagrams, values and/or proportion information. Students should be able to fluidly go from a smaller object to a larger scaled version of the object or from a larger object to a smaller scaled version giving either or both the proportional constant and/or the scale factor.

Solve problems involving scale drawings of geometric figures, such as computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale. 7.G.1.

Scale drawings are diagrams of real measurements with a different unit of measurement, arranged so as to have the same shape as the original they represent. The *scale* describes the relation between the unit of measurement in the drawing and that of the original. Examples of scale models include photographs, doll houses, model trains, architectural designs, souvenirs, maps, and technical drawings for science and engineering. Today with computer image manipulation even in our word processing programs, scaling figures, text, and photos is a common activity. Dynamic visualization tools like Google Earth provide ample real life experience with scale maps and figures.

What exactly is the same and what is different about these scale models and their original counterparts? Linear dimensions on scale models are proportional to the corresponding length on the original: the ratio of any length in the drawing to the corresponding actual length of the original is the *scale* of the drawing, and is the same ratio for any measurement taken on the image. Distortions of a given shape do not count as a scale model. For example, a Barbie doll or cartoon character (like Wreck-it-Ralph) is not proportional to any real human.

Figure 11 (next page) shows a scale drawing of an ant. How long is this ant?

To find the real ant's length we measure the black scale line with a ruler, in order to discover the scale of the drawing. The actual length in your image will depend upon the platform on which you are working, so for this discussion, let us say that the length of that line in the figure is 3 cm, or 30 mm. Thus, the scale for this image is 30 to 1: every linear measurement on the image is 30 times the size of the corresponding measurement of the ant. To answer the question about the actual size of the ant, we have to judge the overall length of the ant's image: this can be a little tricky since we must decide where to place the ruler over the top of the image. Should one start measuring at the antennae and stop at the end of the tail? If one holds the ruler diagonally over the top from head to rear foot, a different measurement results. Nonetheless, it appears that the length of the ant image is about 10 cm, or 100 mm.

Since the scale of image is 30 to 1, every length on the image is 30 times the actual length. Or, the actual length

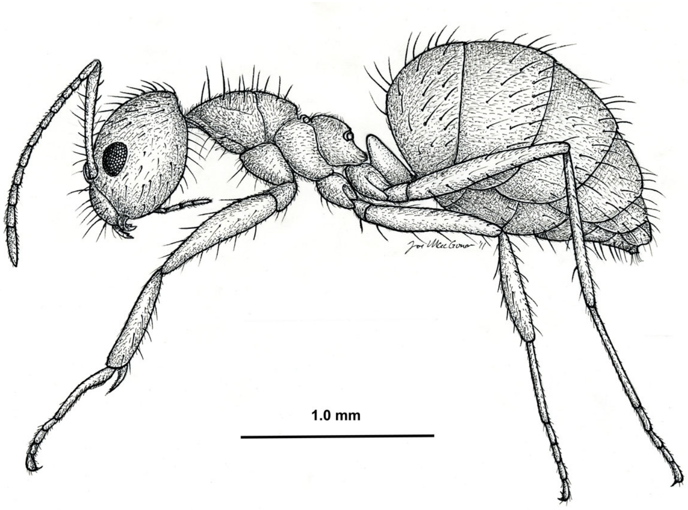


Figure 11: *Nylanderia fulva*, side view of a worker (drawing by Joe MacGown). Image comes from Mississippi Entomological Museum.

is one-thirtieth the length on the image, so that the actual length of the ant is 100/30 = 3 1/3 mm. To clarify the difference between "ratio" and "scale factor,:" the ratio is 30:1, however the scale factor is 1/30th. In other words, one multiplies each length of the original by 1/30 to get the original size of the ant. For example, if the ratio was original: half of 1:$\frac{1}{2}$, then the scale factor would be $\frac{1}{2}$.

Alternatively, we could set up a proportion. The ratio of 1 mm to 3 cm corresponds to the ratio of the ant's actual length in mm to the measurement on the drawing in cm. Hence,

$$\frac{1 \quad \text{mm}}{3 \quad \text{cm}} = \frac{x \quad \text{mm}}{10 \quad \text{cm}}.$$

After solving, we find that $x = \frac{10}{3}$, so the ant is $3\frac{1}{3}$ mm in length. That seems about right for one teeny little ant.

One might also wonder how the artist went about creating this picture. The artist probably looked at an ant specimen under a microscope and drew what he saw as accurately as possible. In real life, biologists often use tiny grids to help them determine the scale of small things they observe under a microscope.

Some important considerations when working with scale drawings:

- The scale of a drawing can be expressed with and without units. For the ant drawing above, the scale (from image to actual) is 3 cm to 1 mm, or 3:0.1 (or 30 to 1: the ant has been magnified 30 times). Maps are scale drawings of actual geography, and the scale may be indicated by a statement: "1 inch = 20 miles," or it might be expressed by labeling an actual line in the drawing with the actual length it represents (as in the Google Maps below). The scale can also be represented as a unit less ratio, such as 1:24,000. If this ratio appears on the map it is telling us that any length on the map represents an actual length that is 24,000 times as long, independent of whether the measurement is made in feet and miles, or in meters and kilometers. Notice that the scale is written with the dimension of the drawing first, and the dimension of the actual last.
- In this sense, scale factors are unitless constants that indicate the relationship between lengths in a scale drawing or model and its real life counterpart. So, a scale factor can be expressed as fraction or even a percentage. If a model is 1% of the real thing (or perhaps a model is 250% of the original), then the percentage expresses the ratio of the length measures in the model to the length measures of the actual object. In this case, any unit may be used. In a model that is 1% of the original, one yard in the original will be 1/100th of a yard in the model. One centimeter in the model will represent 100 centimeters of the original.

- If one uses a scale factor bigger than one, the replica is larger than the original, while if the scale factor is less than one, the model is smaller than the original.
- It is important to keep track of units, if the units are made explicit. A 1:2 ratio or scale factor is different from a ratio of 1 inch: 2 yards.

  A specification of 1:2 ratio without explicit units tells us that the units are the same, so, for example, distances on the scale drawing are half the original distances, so 1 cm corresponds to 2 cm in real life, 1 inch corresponds to 2 inches, and 1 foot corresponds to 2 feet. The 1 inch: 2 yards ratio would be equivalent to a scale of 1:72 since there are 36 inches in a yard. A scale drawing where 1 cm corresponds to 1km does not have a 1:1 scale factor, but rather 1:100,000 since a centimeter is 1/100000 of a kilometer.
- Angles in a scale drawing are the same as the corresponding angle measures in the original.

Some techniques for making scale drawings:

- Overlay a grid, then copy the figure from corresponding grid squares onto a grid of a different size.
- Use proportions to make corresponding side lengths to outline the figure, using the same angle measures from one segment to the next.
- Use computer programs to make scale drawings. (such as Geometer's Sketchpad or Geogebra). If you have a tablet or a smartphone, draw a figure and then expand or contract it in such a way as to have one image be a scale drawing of the other.

When comparing a scale model to the real thing, dimension is also important. As we have seen measures of length scale by the scale factor, but is this the same for area, or, in 3D modeling, volume? Consider this situation:

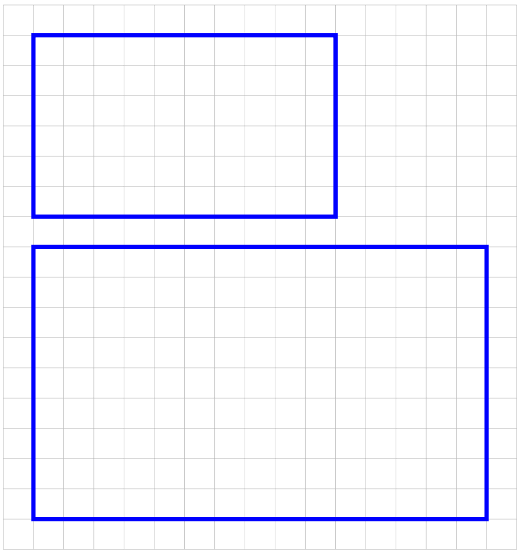


Figure 12

Although the ratio of the top image to the bottom (in Figure 13) is 2:3, the top rectangle contains 60 squares, while the right rectangle contains 135 squares; so in this case the area ratio is 4:9.

Example 4.

Draw a 2x6 rectangle on a piece of graph paper, and then another rectangle in the ratio 1:3. What is the ratio of the areas of these rectangles?

Solution. : The count gives the ratio 12:108, which simplifies to 1:9.

In Figure 13, the scale ratio across all squares is 1:2:3:4:5.. Note that the areas are in the ratio 1:4:9:16:25. These examples support the statement that the scale of area in a scale drawing is the square of the scale of length.

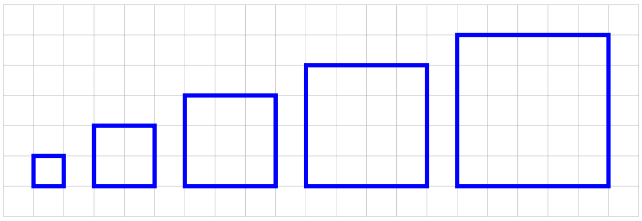
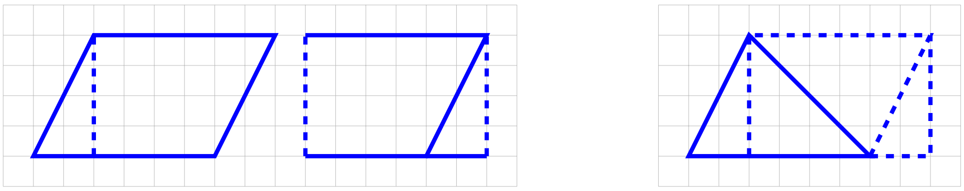


Figure 13

This is true not only for rectangles, but for all figures. For example, since we can move a triangle off a parallelogram to make a rectangle of the same area, it is true of parallelograms, and since a triangle s half a parallelogram it is also true for triangles. In the next section we will show this for the area inside a circle as well.



Area of Parallelogram = 30 Unit Squares

Area of triangle = 12.5 Unit Squares

Figure 14

A fact that comes out of the above discussion of rectangles is this: for two rectangles, if one is a scale drawing of the other, then the ratio of corresponding sides of the rectangles is the same. It is not hard to see that this is also true that two rectangles with the same ratio of sides are scale drawings of each other.

Example 5.

Dave was planning a camping trip to Salina, Utah. He used an online map to find the approximate area of the Butch Cassidy Campground, and to find the distance from the campground to his grandmother's house. His grandmother lives on the northeast corner of 100 East and 200 North in Salina. (See Figure 15). In fact, Dave's grandmother owns that entire block (between 100 and 200 East and 200 and 300 North). What is the area of her property?

Solution. The campground, he reasoned using his index finger tip to measure, looks like it is about 1000 feet by 500 feet, so that's 500,000 square feet in area. He wondered how many acres that would be and quickly looked up the information that 1 acre = 43,560 square feet. Okay, he thought, and punched in 500,000 divided by 43,560 into his calculator, so the campground is about 11.5 acres. Now, to get to Grandma's house, he thought and continued using his finger to measure. It looks like it is over 5000 feet to get to Main Street from the campground entrance, and then probably just over another 1000 feet after that. Recalling that about 5000 feet make a mile, he decides it will be over a mile, but less than a mile and a half , to grandma's house. As for the area of grandmother's property: her block seems to be about 800 feet on a side, so the area of that block s 64,000 square feet, or about 1.5 acres.

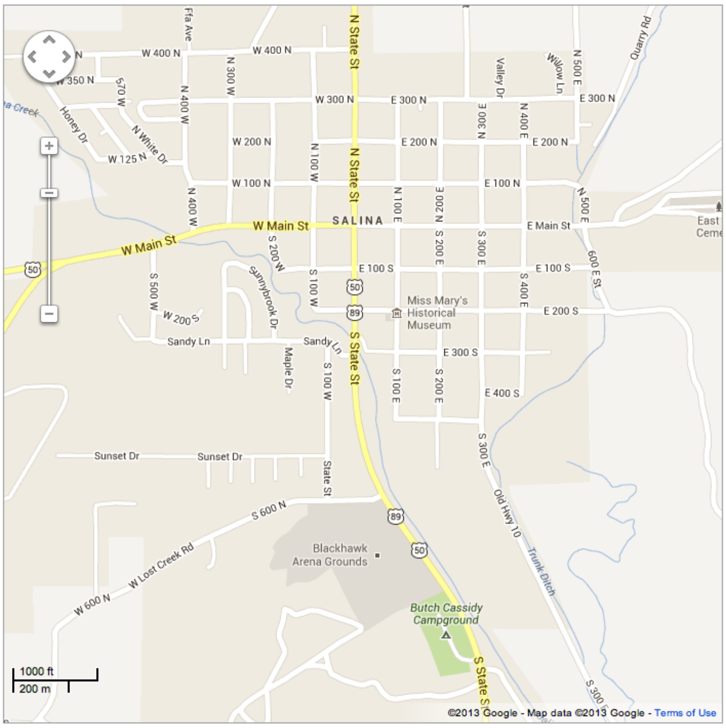


Figure 15: Google maps - Salina, Utah.

Dave then used a different scale map to approximate the distance he'd have to bike to get from the Butch Cassidy Campground to Palisade State Park. This time he decided to print the map and use his ruler. (Figure 16)

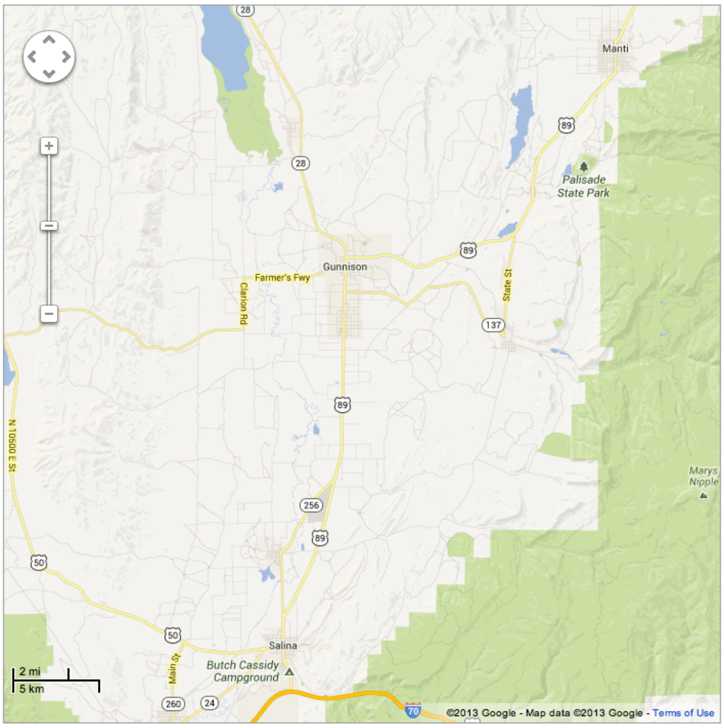


Figure 16: Google maps - Salina, Utah.

Section 5.3. Solving Problems with Circles

In this section circumference and area of a circle will be explored from the perspective of scaling, that is, we start by measuring the diameter and circumference of various circles and noting that the ratio of the circumference to the radius is constant (2π). This leads to discussions about all circles being scaled versions of each other and eventually "developing" an algorithm for finding the area of a circle using strategies used throughout mathematical history. In these explorations, two big ideas are discovered: 1) cutting up a figure and rearranging the pieces preserves area, and 2) creating a rectangle is a convenient way to find area. Additionally, we will connect the formula for finding the area of a circle (πr^2) to finding the area of a rectangle/parallelogram where the base is ? the circumference of the circle and the height is the radius ($A = Cr/2$).

The section will close by applying what was learned to problem situations. Chapter 6 will use ideas of how circumference and area are connected to write equations to solve problems but in this section students should solve problems using informal strategies to solidify their understanding.

Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle. 7.G.4 .

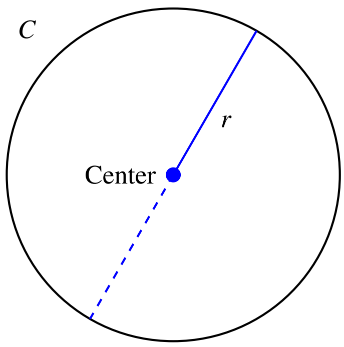


Figure 17

Recall that a *circle* is the set of all points equidistant from a center. We draw circles with a compass by fixing the point of the compass at the center, using a fixed angle at the compass hinge, and rotating the pencil point around to mark the circle. Draw a line segment from the center to a point on the circle; this is a *radius* of the circle. The plural of "radius" is "radii." By definition, all radii are of the same length. If we take any radius, and extend that line segment from the center in the opposite direction of the given radius, we get a *diameter* of the circle (Figure 17).

Observe that any two circles are scale drawings of each other, where the scale factor is the quotient of the lengths of the radii of the two circles. Therefore, if we know the length of the circumference of a circle of radius 1unit, then the length of the circumference of a circle of radius r units is r times that length. Similarly, for area, except that the scale factor of is r^2. We can see that as follows: Draw two circles on a grid, one with twice the radius of the other (Figure 18).

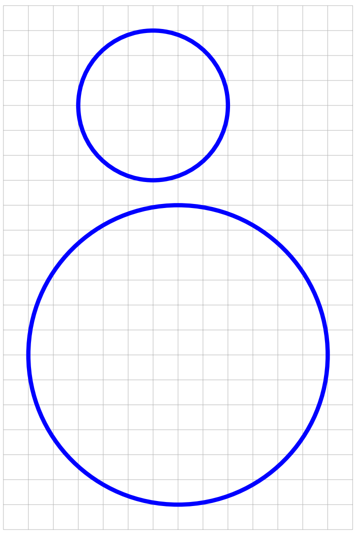


Figure 18

Now count the number of squares of the grid that are more than half inside the circle (use symmetry to make the counting easier). In our case those numbers are 32 for the smaller circle, and 120 for the larger one. This is about 4 times as large. This should be the same for all circles drawn by students: the number for the larger circle is about four times as large as the number for the smaller circle. If we took a different ratio, or a finer grid, the answer will be the same; if the radius is multiplied by some number a, then the area is multiplied by a^2. It follows that, if we know the area of a circle of radius 1unit, then the area of a circle of radius r units is r^2 times that area. So, the ratio of the area of a circle and the square of its radius is a constant, and that constant is the area of circle of radius 1 unit, and is designated by.the greek letter *pi*, written π. Thus we get this formula for the area A of a circle in terms of its radius r: $A = \pi r^2$.

As a side note, methods for computing the area of simple polygons was known to ancient civilizations like the Egyptians, Babylonians and Hindus from very early times in Mathematics. But computing the area of circular regions posed a challenge. Archimedes (287 BC - 212 BC) wrote about using a method of approximating the area of a circle with polygons, as worked through 5.3b Classwork Activity.

The formula $A = \pi r^2$ doesn't help us much until we know the numerical value of π, or at least a good approximation of that numerical value. The exercise we have just done gives us an estimate for π by the count made in the Figure 19: for a circle of radius 6, we counted 120 squares more than half inside circle. So, $120 = \pi 6^2$, from which we get the estimate $120/36 = 3.33$ as an approximate value for π. Now, if we took an even larger circle on the same grid, we'd get a better approximation, and presumably we can get as good an approximation as we want in this way. It would be interesting for each student in the class to make such a calculation, and then take the average as a statistical experimental estimate for π.

Now we turn to circumference of a circle. By the same reasoning (that is, estimating the perimeter of the figure consisting of all the squares counted for area), we can give good evidence that the ratio of the length of its circumference to the length of the radius is constant That this constant is related to π is amazing; we will now demonstrate this using an ancient Egyptian argument.

First, draw a circle of radius r, and let us denote its area by A and the length of the circumference by C. Fill the circle with a coiled rope, starting at the center and circling around until the circumference is reached (Figure 19).

Figure 19

Now place a straight edge along a radius and cut the rope all the way through along that straight edge (see figure 20).

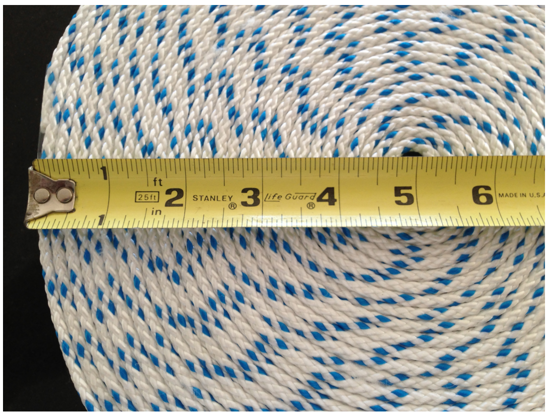


Figure 20

Flatten out the pieces of the rope so that each piece is a horizontal line with the outside piece at the bottom, and the center at the top, to get the isosceles triangle in Figure 21

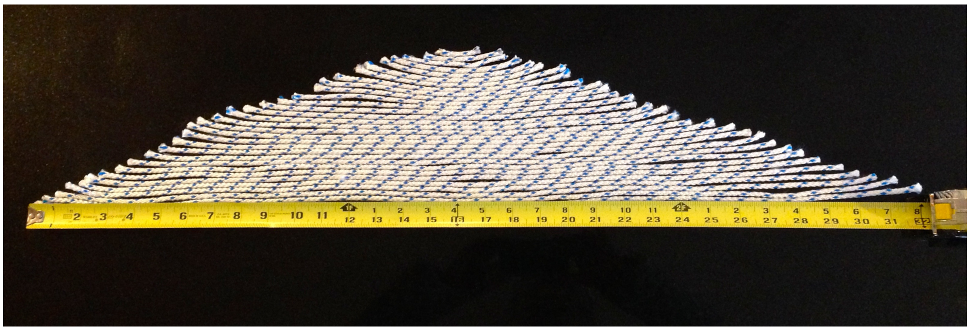


Figure 21

We know that the area of a triangle is one half the product of the base times the height. For this triangle, the base is the circumference of the circle, and the height the radius. Thus the triangle has area $\frac{1}{2}Cr$. But since the area is that filled in by the rope, whether it be coiled in the circle or flattened in the triangle, we conclude that this is the area of the circle:

$$A = \frac{1}{2}Cr\ .$$

Now if we replace A by πr^2, we can solve $\pi r^2 = \frac{1}{2}Cr$ for r to obtain the formula for circumference in terms of radius:

$$C = 2\pi r\ .$$

This "real" construction gives us a way of estimating π,. For thie particular circle in Figures 20-22, the radius is 5 inches, and the circumference (the base of the triangle in the above figure) is 32 inches. From $C = 2\pi r$ we can write $32 = 2\pi(5)$, so we get the estimate 3.2 for the value of π.

Because the rope has substantial thickness, this is a rough estimate, and probably larger than the true value of π. To do better, select a circular disc of some thickness. Measure the length of a radius, call it r. Now take a thin string and wrap it around the circumference of the disc and mark the length that just makes one full loop around the disc. Measure the length of this string, call it C. Then the ratio $C/2r$ is an approximation to the value of π.

The area formula provides another way to evaluate π. Inscribe a regular polygon with n sides, as in Figure 22 (where $n = 10$):

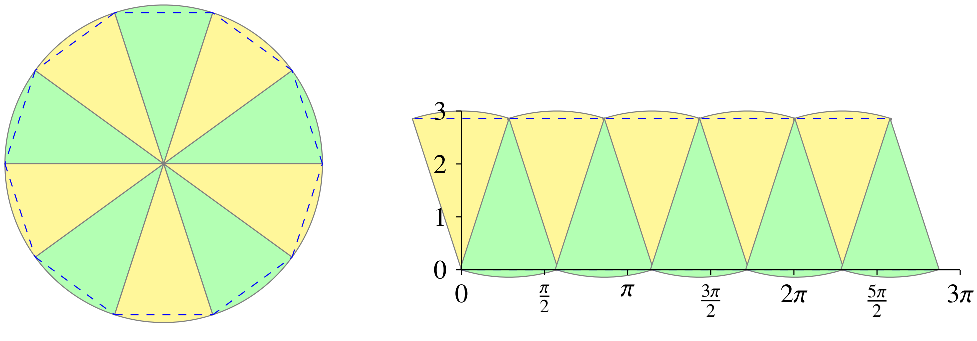


Figure 22

The area of one triangle can be estimated by measuring the base and altitude. Then the area of the polygon is n times that number, and this provides an estimate for the area of the circle.

This second method was used in antiquity by Archimedes, with $n = 16$ to get an approximation for π correct to three figures. Figure 23 also gives us another way of determining the relation $A = \frac{1}{2}Cr$. For each triangle, the area of the triangle is $\Delta) = \frac{1}{2}br$, where Δ is the area of the triangle, and b is its base, and the height is the radius r. Now adding this over all the triangles, amounts to multiplying both sides of the equation by n, the number of triangles, giving us $A = \frac{1}{2}Cr$ for the circle.

Section 5.4. Angle Relationships

The closing section involves applications with angle relationships for vertical angles, complementary angles and supplementary angles. In addition, students will use concepts involving angles to relate scaling of triangles and circles. Practice with the skills learned in this section will be further developed in Chapter 6 when students write equations involving angles.

Use facts about supplementary, complementary, vertical, and adjacent angles to write and solve multi-step problems for an unknown angle in a figure. 7.G.5

First, let us recall some concepts having to do with combining angles.

The sum of the angles $\angle ABC$ and $\angle DEF$ is defined as follows (see Figure 23) : Move (by sliding and rotating) $\angle DEF$ so that the vertices B and E coincide, so that rays BA and ED coincide and so that ray EF is not on the same side of BC as BA. Then the sum of the given angles is the angle $\angle ABF$.

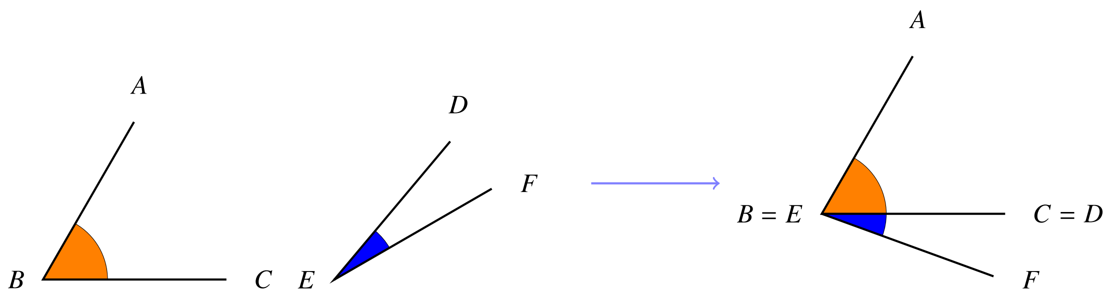


Figure 23

When two lines intersect at a point, four angles are formed. Figure 23 shows line AB intersecting line CD at T. Angles $\angle ATC$ and $\angle CTB$ are *adjacent angles* on a straight line, hence the sum of the measure of these two angles is 180°. Such angles are called *supplementary*. Angles $\angle ATC$ and $\angle BTD$ are called *vertical angles*, in the sense that they are opposing angles at a vertex. Angles $\angle CTB$ and $\angle DTA$ are vertical angles as well. Vertical angles are equal, since they are both supplementary to the same angle. That is, as in Figure 24, $\angle ATC$ and $\angle BTD$ are supplementary to $\angle DTA$, so are equal.

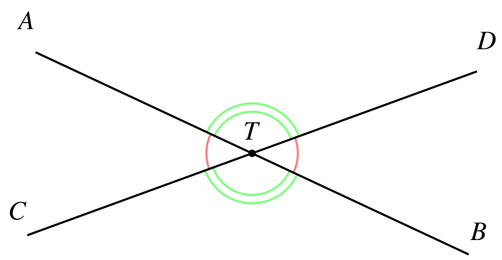


Figure 24

If all the angles in Figure 24 are equal, then the are all of measure 90°; that is, they are all *right* angles. In this case, the lines AB and CD are said to be *perpendicular*.

Two lines are said to be *parallel* if they have no point of intersection. To be clear about this: not just "no point of intersection on our piece of paper," nor in our line of vision, but "no point of intersection" *a*nywhere. This definition supposes that we can imagine the whole plane, infinite in extent in all directions, and can see that the two lines called parallel indeed never intersect. This, then, is a theoretical, rather than an operational definition, and for that reason bothered the mathematicians in Alexander's day (and for the next 2000+ years). The Greeks did the best they could with this problem, and postulated the intersection point in the *Elements of Geometry* in this way.

Given two lines L and L', draw a third line L" that intersects both (see Figure 25), and call the points of intersection A and B.

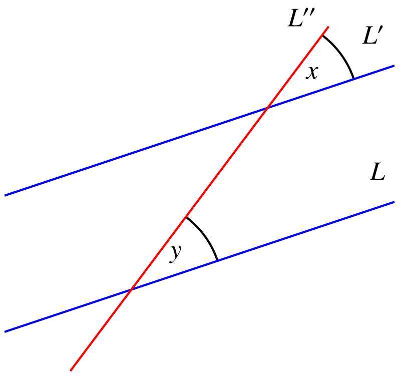


Figure 25

Focus on the "interior angles" x and y. The two lines L and L' are called *parallel* if the angles x and y are of the same measure. In our figure it appears that x and y are of the same measure, so the lines L and L' would be parallel by this definition. It also looks like lines L and L' also will never intersect, but since we cannot see the full extent of the whole plane, we cannot know. So, for the Greek mathematicians of that day, it was clear that the statement that $x = y$ in measure is the same as the statement that "the lines L and L' never meet"- and it was also clear to them that this was a theoretical, not an actual, observation. So, the Parallel Postulate of the *Elements* states this: if the angles at x and y are not the same, then the lines L and L' eventually meet.

Example 6.

The sum of the angles of a triangle is 180° (see Figure 26). To see this, draw a triangle, cut out the angles and put all the vertices together, so that the angles are adjacent and not overlapping. You will get a straight angle. If everyone in the class does this, they will all get the same result: a straight angle. Keep in mind that this is very convincing evidence, but does not explain why this is true.

Figure 26

Now, let us return to Figure 25 and suppose the lines L and L' are not parallel; that is, they intersect at some point V. Figure 27 envisions this; the lines may have extended for millions of miles before arriving at that intersection. For this reason the lines are dashed: we don't know how far away V is, but we are imagining our paper to be large enough to reach to V. Since the angle at V has to have positive measure, the angles at A and B (of the triangle ΔAVB) have measures that add up to less than 180°, so those angles cannot be supplementary. Since the angle $\angle ABV$ and x are supplementary, x an y do not have the same measure.

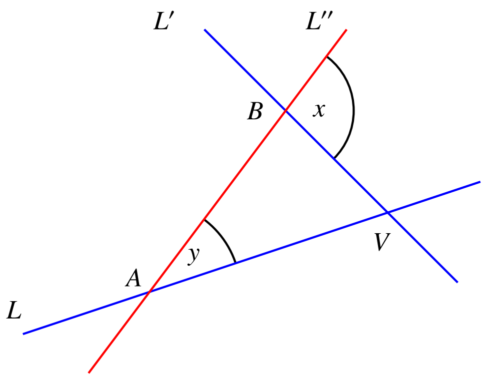


Figure 27

So, it follows follows that if the measures of the angles x and y are equal, then the lines L and L' can never intersect. For if they did intersect we'd have a picture just like that of the lines L and L', and we just saw that if the lines intersect, the interior angles cannot be equal.

Example 7.

Draw a circle and label the center O. In the circle draw a diameter, labeling the endpoints A and B. Now select a point C on the circle, different from A and B and draw the triangle ABC. See Figure 28:

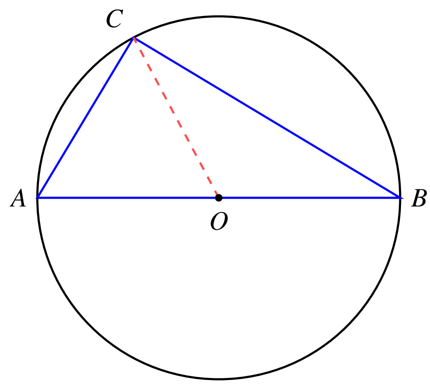


Figure 28

With a protractor find the measure of $\angle ACB$. It is 90°, so $\angle ACB$ is a right angle.

For an explanation of this, draw the ray from O to C. Lines AO,BO, CO are all of equal length since they are all radii of the circle. In particular, triangles AOC and BOC are isosceles triangles, and so the base angles are equal (a property of isosceles triangles). That tells us that the measure of $\angle ACO$ is also x and the measure of $\angle BCO$ is also y, so that the measure of $\angle ACB$ is $x + y$. But, since the sum of the measures of the angles of a triangle is 180°, we have

$$x + \angle ACB + y = x + (x + y) + y = 180 ,$$

so $2(x + y) = 180$, or $x + y = 90$ and thus $\angle ACB$ has measure 90°.

Now, the student should remember, from 5th grade, the fact that for an isosceles triangle (a triangle with two sides equal), the base angles are equal. This can be demonstrated as follows: draw an isosceles triangle AVB, as in the following figure, with the lengths of AV and BV equal. Now fold the triangle along a line that goes through the vertex V, so that the line segment AB folds over onto itself. The student will see that in fact the triangle on one side is perfectly superimposed on the triangle on the other side, so that the angle at A lies directly over the angle at B, and so they are of equal measure.

Example 8.

Here is another interesting fact that follows from the basic fact about the sum of the angles of a triangle. Draw a circle and select three points A, V, B on the circle and draw the angle AVB. Suppose that the center of the circle O lies inside $\angle AVB$, as in the figure below. Draw the line segments AO and BO. Then the measure of $\angle AVB$ is half the measure of $\angle AOB$.

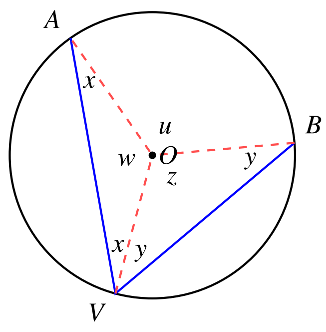


Figure 29

Solution. To see this, label the measures of the angles with the letters x, y, z, w, u as in the diagram. Because the lines AO, VO, BO are all radii, and thus of the same lengths, angles with the same letter indeed do have the same measure. Now, look at all the triangles involved to get:

$$2x + w = 180^\circ\ , \qquad 2y + z = 180^\circ\ , \qquad u + w + z = 360^\circ\ .$$

Add the first two equations to get

$$2x + 2y + w + z = 360^\circ$$

and use the third equation to get $2x + 2y + 360 - u = 360^\circ$, and now conclude that $u = 2(x + y)$. But u is the measure of $\angle AOB$ and $x + y$ is the measure of $\angle AVB$.

Example 9.

In the above activity, we assumed that the given angle had its rays on opposite sides of the center of the triangle. However, the statement is still true for any angle with vertex on the circle. Students may want to try to figure out why.

Example 10.

In Figure 30, the measures of two angles are given. Find the measures of the remaining angles.

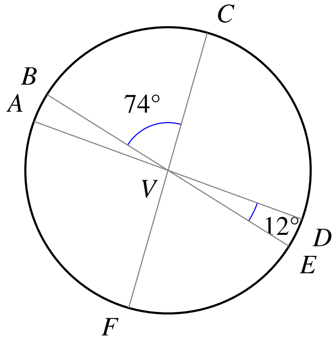


Figure 30

SOLUTION. The sum of angles $\angle BVC$, $\angle CVD$ and $\angle DVE$ is a straight angle, so has 180°. This tells us that $74+\angle CVD+12 = 180$, giving us the measure of $\angle CVD$, 94°. Each of the other angles is the vertical angle associated to one of these, so that gives us all the measures.

EXAMPLE 11.

Find the measures of all the angles in Figure 31.

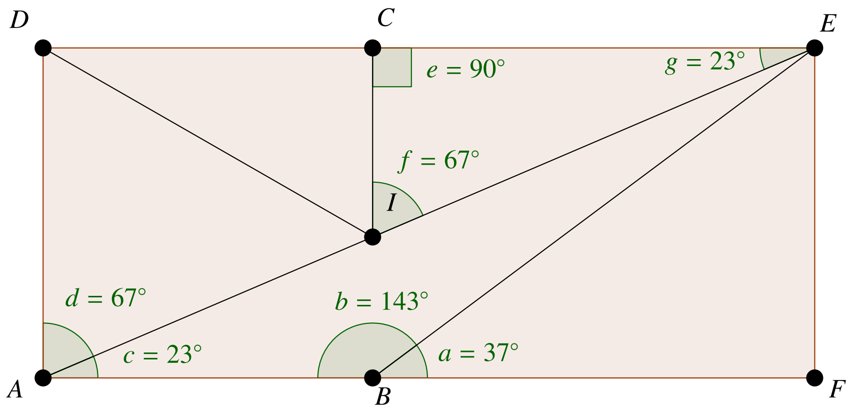


Figure 31

SOLUTION. We see a complex geometric diagram that shows many angle relationships, so we should proceed carefully using the basic angle relationships: vertical angles are equal, supplementary angles add to 180°, and the sum of the angles of a triangle is 180°, We can see that $\angle FBE$ and $\angle EBA$ are supplementary. $\angle BAI$ and $\angle IAD$ are complementary. Two angles do not need to be adjacent to be considered supplementary or complementary. In this figure, $\angle EIC$ and $\angle CEI$ are complementary. One way to know this is by measuring. Another way, is to recognize that $\triangle ICE$ is a right triangle, and apply the fact that the sum of angles in a triangle is 180°. Since $\angle ICE$ is a right angle, the sum of the other two must be 90° so that the total is 180°. With such a figure, students can be prompted to use the triangle sum theorem to fill in additional angle measures, and name additional pairs of supplementary and complementary angles.

Chapter 6
Real World Equations and Inequalities

In this chapter we will use the geometric relationships that we explored in the last chapter and combine them with the algebra we have learned in prior chapters. First we recall the methods developed in (the first section of) Chapter 3, but without the bar models, focusing on algebra as an extension of the natural arithmetic operations we have been performing until now in solving problems. We will then focus on how to solve a variety of geometric applications and word problems. We will also encounter inequalities and extend our algebraic skills for solving inequalities. By this chapter's end, we will be able to set up, solve, and interpret the solutions for a wide variety of equations and inequalities that involve rational number coefficients.

This chapter brings together several ideas. The theme throughout however is writing equations or inequalities to represent contexts. In the first section students work with ideas in geometry and represent their thinking with equations. Also in that section students solidify their understanding of the relationship between measuring in one-, two-, and three-dimensions. In the second section, students will be writing equations for a variety of real life contexts and then finding solutions. The last section explores inequalities. This is the first time students think about solutions to situations as having a range of answers.

In Chapter 3 students learned how to solve one-step and simple multi-step equations using models. In this chapter students extend that work to more complex contexts. In particular they build on understandings developed in Chapter 5 about geometric figures and their relationships. Work on inequalities in this chapter builds on Grade 6 understandings where students were introduced to inequalities represented on a number line. The goal is Grade 7 is to move to solving simple one-step inequalities, representing ideas symbolically rather than with models.

Throughout mathematics, students need to be able to model a variety of contexts with algebraic expressions and equations. Further, algebraic expressions help shed new light on the structure of the context. Thus the work in this chapter helps to move students to thinking about concrete situations in more abstract terms. Lastly, by understanding how an unknown in an expression or an equation can represent a "fixed" quantity, students will be able to move to contexts where the unknown can represent variable amounts (i.e. functions in Grade 8.)

Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities. 7.EE.4

Consider the sentences $9 + 4 = 13$ and $8 + 6 = 12$. The first sentence is true, but the second sentence is false. Here is another sentence: $\diamond + 9 = 2$. This sentence is neither true nor false because we don't know what number the symbol $\diamond$ represents. If $\diamond$ represents -6 the sentence is false, and if $\diamond$ represents -7 the sentence is true. This is called an *open sentence*.

A symbol such as $\diamond$ is referred to as an *unknown* or a *variable*, depending upon the context. If the context is a specific situation in which we seek the numeric value of a quantity defined by a set of conditions, then we will say it is an "unknown." But if we are discussing quantifiable concepts (like length, temperature, speed), over a whole range of possible specific situations, we will use the word "variable."

So, for example, if I am told that Maria , who is now 37, three years ago was twice as old as Jubana was then, then I would write down the equation $37 - 3 = 2(J - 3)$, where J is the unknown age of Jubana. But if I write that $C = 2\pi r$ for a circle, C and r are the measures of circumference and radius (in the same units) for *any* circle. In this context, C and r are “variables.”

Various letters or symbols can be used, such as $\diamond, x, y, a, b$, and c. Variables are symbols used to represent any number coming from a particular set (such as the set of integers or the set of real numbers). Often variables are used to stand for quantities that vary, like a person’s age, the price of a bicycle, or the length of a side of a triangle. But, in a specific instance, if we write $\diamond + 9 = 2$, $\diamond$ is an unknown, and the number -7 makes $\diamond + 9 = 2$ true, so is a correct value of the unknown, and is called a *solution* of the open sentence.

Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related. 7.EE.a2

In Chapter 3, we moved seamlessly from pictorial models of arithmetic situations to algebraic formulations of those models, called expressions, without having said what an expression is. Let’s do that now: : an *expression* is a phrase consisting of symbols (representing unknowns or variables) and numbers, connected meaningfully by arithmetic operations. So $2x - 3(5 - x)$ is an expression as is $(4/x)(x^2 + 3x)$.

It is often the case that different expressions have the same meaning: for example $x + x$ and $2x$ have the same meaning, as do $x - x$ and 0. By *the same meaning* we mean that a substitution of any number for the unknown x in each expression produces the same numerical result. We shall call two expressions *equivalent* if they have the same meaning in this sense: any substitution of a number for the unknown gives the same result for both expressions.

Since checking two expressions for every substitution of a number will take a long time, we need some rules for equivalence. These are the laws of arithmetic: $2x + 6$ and $2(3 + x)$ are equivalent because of the laws of distribution and commutativity. In the same way, $2x + 5x$ is equivalent to $7x$; $-2(8x - 1)$ is equivalent to $2 - 16x$, and so forth. Reliance on the laws of arithmetic is essential: to show that two expressions have the same meaning, we don’t/can’t check every number; it suffices to show that we can move from one expression to the other using those laws. Also, to show that, for example $3 + 2x$ and $5x$ are not equivalent, we only have to show that there is a value of x that, when substituted, does not give the same result. So, if we substitute 1 for x we get 5 and 5. But what if we substitute 2 for x: we get 7 and 10. The expressions are not equivalent.

To show that two expressions are equivalent, we must show how to get from one to the other by the laws of arithmetic. However, to show two expressions are not equivalent, we need only find a substitution for the unknown that gives different results for the two expressions.

Open sentences that use the symbol ‘=’ are called *equations*. An equation is a statement that two expressions on either side of the ‘=’ symbol are equal. The mathematician Robert Recorde invented the symbol to stand for ‘is equal to’ in the 16th century because he felt that no two things were more alike than two line segments of equal length. These equations involve certain specific numbers and letters. We refer to the letters as unknowns, that is they represent actual numbers which are not yet made specific. Indeed, the task is to find the values of the unknowns that make the equation true. If an equation is true for all possible numerical values of the unknowns (such as $x + x = 2x$), then the equation is said to be an equivalence. It is an important aspect of equations that the two expressions on either side of the equal sign might not actually always be equal; that is, the equation might be a true statement for some values of the variables(s) and a false statement for others.

For example, $10 + 0.02n = 20$ is true only if $n = 500$; and $3 + x = 4 + x$ is not true for any number x; and $2(a + 1) = 2a + 2$ is true for all numbers a. A solution to an equation is a number that makes the equation true when substituted for the variable (or, if there is more than one variable, it is the number for each variable). An equation may have no solutions (e.g. $3 + x = 4 + x$ has no solutions because, no matter what number x is, it is not true that adding 3 to x yields the same answer as adding 4 to x. An equation may also have every number for a solution (e.g. $2(a + 1) = 2a + 2$). An equation that is true no matter what number the variable represents is called an identity, and the expressions on each side of the equation are said to be equivalent expressions. So,, $2(a + 1)$ and $2a + 2$ are equivalent expressions.

Statements involving the symbols '>', '<', '≥', '≤' or '≠' are called *inequalities*. Let's review the meanings of these symbols:

- $x > 3$ means "x is greater than 3."
- $7 < 4$ mens "7 is less than 4." (Note that these are simply statements, with truth or falsity to be determined.)
- $x \geq y$ means "x is greater than or equal to y."
- $x \leq x^2$ means "x is less than or equal to x^2."
- $a \neq b + c$ means "a is not equal to $b + c$."

To *solve* an equation or inequality means to determine whether or not it is true, or for what values of the unknown it is true. Once found, those values are called the *solutions* of the problem.

Example 1.

Find the solutions of these open sentences by inspection. How many solutions are there?

a. $\heartsuit - 4 = 1$.
b. $x + \frac{1}{4} = \frac{7}{4}$.
c. $5x = 12$.
d. $2x = 0$.
e. $0x = 5$.
f. x is a whole number and $x < 4$.
g. $x = 2x - x$.

Solution.

a. $\heartsuit = 5$: one solution;

b. $x = \frac{6}{4} = \frac{3}{2} = 1\frac{1}{2}$: one solution;

c. $x = \frac{12}{5} = 2\frac{2}{5}$: one solution;

d. $x = 0$: one solution;

e. any number x would still make the left side equal zero; and $0 \neq 5$, therefore no solutions exist;

f. the solutions are 1, 2, and 3: three solutions;

g. Any value of x makes this sentence true, that is, every number is a solution.

Variables, equations, and inequalities empower us to write down ideas in concise ways.

EXAMPLE 2.

Sofia is in third grade and loves mathematics. She was thinking about numbers one day, and she wrote

$$2+2=4+4=8+8=16+16=32+32$$

on her paper and was eager to continue. Apollo was impressed, but confused. "Wait, something's wrong," he told her. " 2 + 2 is **not** equal to 4 + 4. " Sofia agreed and corrected her writing like so

$$2+2=4. \quad 4+4=8. \quad 8+8=16. \quad 16+16=32. \quad \text{etc...}$$

An equation is a sentence. The symbol '=' means 'is equal to,' and if you run all your thoughts together, you may say something false.

Writing mathematics has a grammar of its own. Just like a sentence has punctuation marks, mathematics makes use of *grouping symbols* to help the reader understand the meaning. Equality and inequality symbols are grouping symbols. They divide an equation or inequality into a left hand side (LHS) and a right hand side (RHS). Parentheses and fraction bars are also used as grouping symbols.

To better understand what is meant by *expression*, *equation*, and *inequality* examine the examples in the table below. Note that equations and inequalities are statements, while expressions are more like mathematical phrases.

expressions	equations	inequalities
$\frac{(20-2)}{6}$	$\frac{(20-2)}{6}=3$	$\frac{(20-2)}{6}>0$
$5x-3$	$5x-3=22$	$5x-3\leq 22$
$x^2+2x-21$	$x^2+2x-21=0$	$5\neq -5$
$2l+2w$	$A=\frac{1}{2}bh$	$\lvert AB\rvert+\lvert BC\rvert>\lvert AC\rvert$
$\frac{\pi r^2}{2\pi r}$	$m(\angle 1)+m(\angle 2)=90°$	$\frac{a}{b}<\frac{c}{d}$

As with expressions, equations are said to be *equivalent* if they have the same meaning. For expressions, this meant that substitution of a number for the unknown in each expression gives the same answer. For equations, it is the same, but as an equation represents a statement, the criterion is *true* or *false*: so, the test is this: if the substitution of a number for the unknown in the two equations always gives the same answer to the question, "True or False," then the equations are equivalent. Another way of putting this is this: a *solution* of an equation is a number that, when substituted for the unknown, gives the response "true." So, to make the definition more precise, two equations are*equivalent* if they have the same set of solutions.

As with expressions, this is an impossible criterion to apply: we cannot test every number. So, we look for laws of arithmetic that do not change the solution set of an equation. For example, $2x=10$ and $x=5$ are equivalent equations: because one equation is double the other, so they have the same solution set. Also, $3(x+9)=72$ and $4x-10=50$ are equivalent, because they have the same solution ($x=15$).

EXAMPLE 3.

Write down the sequence of laws of arithmetic the take us from one equation to the other.

SOLUTION. We note first that the following equations are equivalent

$$\begin{aligned} 3x+6 &= 15 && (1)\\ 3x+8 &= 17 && (2)\\ 3x &= 9 && (3)\\ 12x &= 36 && (4)\\ x &= 3 && (5) \end{aligned}$$

The above five equations are all equivalent because they all have the same solution ($x = 3$). But we can also see that the equations are equivalent because they are related by operations that do not change the solution set:

- If we add 2 to both sides of equation (1), we obtain equation (2).
- If we subtract 8 from both sides of equation (2), we obtain equation (3).
- If we multiply both sides of equation (3) by 4, we obtain equation (4).
- If we divide both sides of equation (4) by 12, we obtain equation (5).

This example illustrates the most important operations on equations that do not change the solution set. They are:

1. By adding (or subtracting) the same number on both sides of an equation, the new equation is equivalent to the original equation.

2. Multiplying (or dividing) the same *nonzero* number on both sides of an equation, the new equation is equivalent to the original equation.

Note that when we 'multiply or divide both sides of an equation by a number,' we must apply that operation *to every term* on both sides of the equation. For example: solve

$$3x + 9 = 87 .$$

Our goal is an equation of the form "$x = \ldots$," so first let us divide both sides by 3 to get:

$$x + 3 = 29 ;$$

every term in the equation has been divided by 3. Subtract 3 from both sides to get $x = 26$. We also could have first subtracted 9 from both sides to get $3x = 78$, and then divided by 3. The order of operations does not matter; what is important is that we employ only operations that can be reversed: if we divide all terms in an equation by 3, we can go back to the original equation by multiplying all terms by 3. If we subtract 9 from both sides of the equation, we can go back by multiplying by 9.

EXAMPLE 4.

Explain why it is important to say *nonzero* number in the second rule above, but not in the first rule above.

SOLUTION. Suppose you add or subtract 0 from both sides of an equation. This doesn't change the equation at all. Adding 0 is the reverse of subtracting 0, so these operations can be undone.

Suppose now that you multiply both sides of an equation by 0. For example, if we start with $2x-7 = 15$, multiplying by 0 gives us $0 = 0$. Certainly a simplification, but not a valuable one: there is no way to go from $0 = 0$ back to $2x - 7 = 15$.

Dividing by zero is not allowed because it simply doesn't make sense. Recall that division can be thought of as the inverse of multiplication. But we just saw that if we multiply all terms of *any* equation by 0, we get to $0 = 0$. There is no way of reversing this: to get from the equation $0 = 0$ to any equation.

Finally, if an expression in an equation is replaced by an equivalent expression, then the equations are equivalent. As an example:

$$3(x+2) = 15 \tag{6}$$
$$3x+6 = 15 \tag{7}$$

are equivalent equations.

Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies. 7.EE.b3.

To 'solve' or 'find solutions' of a given equation, perform the previously mentioned operations to obtain equivalent equations until the *unknown* is alone on one side of the equation. We also call this process 'isolating' the unknown on one side of the equation. It can be helpful to simplify expressions on each side of an equation before or while solving the equation. Let's look at some examples.

EXAMPLE 5.

Solve $4x - 7 = 9$.

SOLUTION.

$$\begin{aligned}
4x - 7 &= 9 \\
4x - 7 + 7 &= 9 + 7 \quad \text{(Add 7 to both sides)} \\
4x &= 16 \\
\frac{1}{4}(4x) &= \frac{1}{4}(16) \quad \text{(Multiply both sides by } \frac{1}{4}\text{)} \\
x &= 4
\end{aligned}$$

Therefore, the solution is 4.

Every time we write a new line, we have an algebraic reason for doing so. Recall that multiplying by $\frac{1}{4}$ is the same as dividing by 4.

In chapter 3 we learned various properties of addition and multiplication. Here we study examples with focus on how those properties appear in each step as we solve equations.

Example 6.

Solve $\frac{3}{5}\left(x-\frac{5}{6}\right)=\frac{7}{4}$.

Solution.

$$\begin{aligned}
\frac{3}{5}\left(x-\frac{5}{6}\right) &= \frac{3}{4} \\
\frac{5}{3}\cdot\frac{3}{5}\left(x-\frac{5}{6}\right) &= \frac{5}{3}\cdot\frac{3}{4} \quad \text{(Multiply both sides by } \frac{5}{3}\text{)} \\
\left(x-\frac{5}{6}\right) &= \frac{5}{4} \\
x-\frac{5}{6}+\frac{5}{6} &= \frac{5}{4}+\frac{5}{6} \quad \text{(Add } \frac{5}{6} \text{ to both sides)} \\
x &= \frac{15}{12}+\frac{10}{12}=\frac{25}{12}
\end{aligned}$$

Therefore, the solution is $\frac{25}{12}$ or $2\frac{1}{12}$.

Another solution method:

$$\begin{aligned}
\frac{3}{5}\left(x-\frac{5}{6}\right) &= \frac{3}{4} \\
\frac{3}{5}x-\frac{3}{5}\cdot\frac{5}{6} &= \frac{3}{4} \quad \text{(Apply the distributive property on the LHS.)} \\
\frac{3}{5}x-\frac{1}{2} &= \frac{3}{4} \quad \text{(Simplify the expression } \frac{3}{5}\cdot\frac{5}{6}\text{)} \\
\frac{3}{5}x-\frac{1}{2}+\frac{1}{2} &= \frac{3}{4}+\frac{1}{2} \quad \text{(Add } \frac{1}{2} \text{ to both sides.)} \\
\frac{3}{5}x &= \frac{3}{4}+\frac{2}{4} \\
\frac{3}{5}x &= \frac{5}{4} \\
\frac{5}{3}\cdot\frac{3}{5}x &= \frac{5}{3}\cdot\frac{5}{4} \quad \text{(Multiply } \frac{5}{3} \text{ to both sides.)} \\
x &= \frac{25}{12}
\end{aligned}$$

Again the solution is $\frac{25}{12}$ or $2\frac{1}{12}$.

Example 7.

This process works even with tricky examples. Here we have positive and negative decimal coefficients $0.2x-0.4=-3.4$.

Solution.

$$\begin{aligned}
0.2x-0.4 &= -3.4 \\
0.2x-0.4+0.4 &= -3.4+0.4 \quad \text{(Add 0.4 to both sides.)} \\
0.2x &= -3.0 \\
\frac{1}{0.2}(0.2x) &= \frac{1}{0.2}(-3.0) \quad \text{(Multiply both sides by } \frac{1}{0.2}\text{.)} \\
x &= \frac{-3.0}{0.2}=-\frac{3.0}{0.2}=-\frac{30}{2}=-15
\end{aligned}$$

Therefore, the value of x is -15.

In the next to last step, we could have said we divided both sides by 0.2. This is the same as multiplying both sides by $\frac{1}{0.2}$, the multiplicative inverse of 0.2. Notice also that $\frac{1}{0.2} = \frac{10}{2} = 5$. So, another way to finish solving this equation is by multiplying both sides by 5.

Here is another solution method for this same example:

$$\begin{aligned}
0.2x - 0.4 &= -3.4 \\
2x - 4 &= -34 \quad \text{(Multiply both sides by 10, now the decimals are removed.)} \\
2x &= -30 \quad \text{(Add 4 to both sides.)} \\
\frac{2x}{2} &= \frac{-30}{2} \quad \text{(Divide both sides by 2.)} \\
x &= -15
\end{aligned}$$

There are many ways to get to the right solution. It could be advantageous to multiply each side of an equation by a number just to remove fractions and decimals.

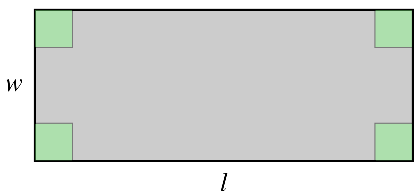
An equation of the form $px + q = r$, where p, q, and r are any numbers, is called a *first order equation*. All of the examples illustrated here are equations that can be written in this form (with $p \neq 0$).

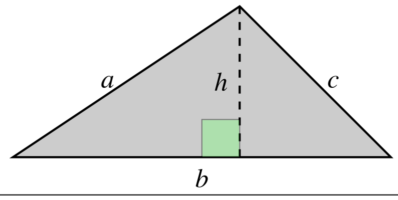
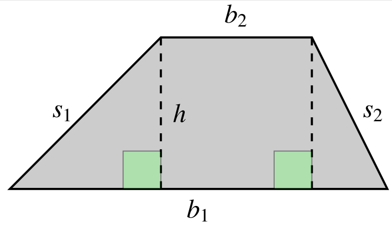
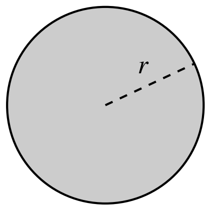
A *formula* tells us information about how different variables relate to each other (the plural form of 'formula' is 'formulae'). For example, the perimeter of a polygon is the sum of the lengths of its sides. Perimeter is usually denoted by the letter P. For a triangle of side lengths a, b, c, we express this by the formula $P = a + b + c$. So, if we are given a triangle of side lengths 19, 7, 21 units, we calculate perimeter as follows:

$$\begin{aligned}
P &= a + b + c, \quad a = 19, \quad b = 7, \quad c = 21 \\
P &= 19 + 7 + 21 = 47
\end{aligned}$$

Look over the following table of mathematical formulae for the areas and perimeters of geometric objects. Remember that if length is given in certain units (ft., cm., ...), then perimeter is measured in the same units, while area is measured in square units (sq. ft., sq. cm., ...).

For example, if we have a rectangular lot that measures 72 yards in length and 72 yards in width, then to fence in the whole lot we need to calculate the perimeter and use the formula $P = 2l + 2w$, with $l = 72$ yards and $w = 40$ yards: $P = 2(72) + 2(40) = 224$ yards. To find the area of this lot we use the formula $A = l \cdot w$, so it has area $A = 72 \cdot 40 = 2880$ sq. yds.

	$P = 2l + 2w$	$A = l \cdot w$

	$P = a + b + c$	$A = \frac{1}{2}b \cdot h$
	$P = b_1 + b_2 + s_1 + s_2$	$A = \frac{1}{2}(b_1 + b_2)h$
	$C = 2\pi r$	$A = \pi r^2$

Section 6.1. Write and Solve Equations to Find Unknowns in Geometric Situations

This section builds upon what students learned about geometric relationships in chapter 5 and in earlier grades. We begin by applying the skills used in solving equations, to writing and solving one-step and multi-step equations involving finding missing measures of unknown values in contexts which involves various angle relationships with triangles, areas, perimeters, circles and scaling. In particular, we will pay close attention to the relationship between the structure of algebraic equations and expressions and the contexts they represent.

Use facts about supplementary, complementary, vertical, and adjacent angles to write and solve multi-step problems for an unknown angle in a figure. 7.G.5

In the last chapter, we learned many relationships among angles that can be expressed with equations.

- The sum of the measures of interior angles in a triangle is 180°.
- Vertical angles have equal measure.
- Complementary angles add up to 90°.
- Supplementary angles add up to 180°.

The use of algebraic language can help us to write relationships and solve problems. Along with algebraic language, angles are relevant to the world around us. Suppose we knew a few things about the angles in this library, and wanted to know other angles, that is, we explore how other measures can be used to determine what missing angle measure is. As in the last examples of chapter 5, we will set up and solve equations for some of the angles, knowing others.

Example 8.

In the library shown in figure 1, the pitch of the roof is 22° and the angle between the joist and the roof support beam is 38°. Find the measure of the remaining angles of the wooden support structure.

Figure 1

Solution. We have used some technical language in phrasing this problem, so let's take a moment to clarify the language. Additionally, a scale drawing of the roof support structure connects the language with a visual.

The *pitch* of the roof is the angle it makes with the horizontal. So we have put a horizontal hashed line at the peak of the roof and indicate the angle with measure 22°. A *roof support beam* is a beam from a wall of the structure that makes a diagonal with the roof; in our case, this is the beam represented by the line segment AB. A *joist* is a horizontal floor beam (even if there is no floor); designated in our diagram by the segment AC, giving the angle measure 38° as labeled in Figure 1. Finally, although it has not been made explicit, we assume that the walls are perfectly vertical; in particular, the line segment AD is vertical.

Since AD is vertical and the hashed line is horizontal, the angle $\angle ADB$ is complementary to the 22° angle, and so $\angle ADB = 90 - 22 = 68°$. Additonally, $\angle DAB$ is complementary to $\angle BAC$ whose measure is 38°. We conclude that $\angle DAB$ has measure 52°. Given that the sum of the angles of a triangle is 180°, we find the measure of $\angle DBA$:

$$\angle DBA = 180 - (68 + 52) = 60° \ .$$

Example 9.

Determine if the statement below is *always, sometimes or never* true. It if is sometimes true, give an example when it is true and when it is false. If is it *never* true, give a counter example.

a. Adjacent angles are also supplementary angles.

b. Vertical angles have the same measure.

Solution. First, we note that the terms counter example and non-example are interchangeable, that is, they represent an example that disproves a proposition. Examining examples and non-examples can help students understand definitions. Constructing an argument to disprove a statement only requires one counter example, while constructing an argument to "prove" something is more involved. In other words, one affirmative example does not prove a statement. In Grade 7 attention to precision in making statements is an important first step towards building arguments. So, for part b, in making the determination that it is always true, we look to explain why the statement is true, looking for statements that build on understanding of supplementary angles and transitivity.

a. Sometimes true. Counter example: the adjacent angles in Figure 1, $\angle CAB$ and $\angle BAD$.

b. True. In Figure 2 angles $\angle ATC$ and $\angle BTD$ are vertical angles, in the sense that they are opposing angles at a vertex. Angles $\angle CTB$ and $\angle DTA$ are vertical angles as well. Vertical angles have equal measure, since they are both supplementary to the same angle. That is, in Figure 2, $\angle ATC$ and $\angle BTD$ are supplementary to $\angle DTA$, so they must have the same measure.

Additionally, in Figure 2, both angles $\angle ATD$ and $\angle CTB$ are supplementary to $\angle DTB$, therefore they also are equal.

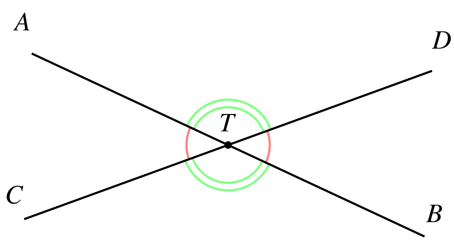


Figure 2

Example 10.

Find the values of x and y in Figure 3.

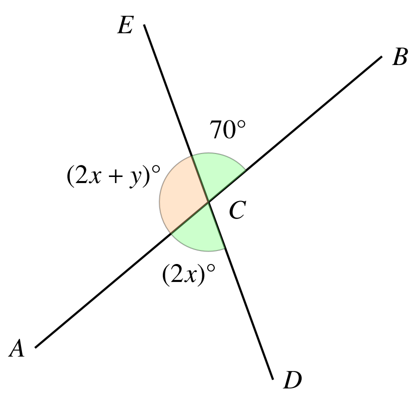


Figure 3

Solution.

$$\begin{aligned}
2x &= 70^\circ && \text{(Vertical angles are congruent)}\\
x &= 35^\circ && \text{(Multiply both sides by} 1/2)\\
2x + y + 70 &= 180^\circ && \text{(Supplementary angles add to 180)}\\
2x + y &= 110^\circ && \text{(Add the opposite of 70 to both sides)}\\
70 + y &= 110^\circ && \text{(since } 2x \text{ is} 70)\\
y &= 40^\circ && \text{(Add the opposite of 70 to both sides)}\ .
\end{aligned}$$

Example 11.

The perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width?

Solution. . Since we know that the perimeter of a rectangle is equal to 2 times the length plus 2 times the width, we can start by writing the formula $P = 2l + 2w$. In this example the perimeter is 54 cm and the length is 6 cm. Placing the information into our equation yields a new equation with just one unknown quantity, w:

$$54 = 2(6) + 2w$$

Now, we can proceed with solving the equation. We get:

$$\begin{aligned}
54 &= 12 + 2w\\
54 - 12 &= 12 + 2w - 12\\
42 &= 2w\\
\frac{42}{2} &= \frac{2w}{2}\\
21 &= w\ .
\end{aligned}$$

We have found that $w = 21$, so the width of the rectangle must be 21 cm.

Example 12.

A trapezoid has a perimeter of 47 cm. and an area of 132 sq. cm. The longer of the two parallel sides has length 13 cm and the height of the trapezoid is 11 cm. What is the length of the shorter of the two parallel sides?

Solution. We refer to the table of formulae above: the perimeter of the trapezoid is given by the formula

$$P = b_1 + b_2 + s_1 + s_2$$

where the b's refer to the parallel sides, and the s's refer to the other sides. Putting information into the formula gives us the equation

$$47 = b_1 + 13 + s_1 + s_2,$$

and we want to find b_1. But we don't know s_1 and s_2; what we do know is the height of the parallelogram, but that does not tell us the length of the sides (draw several parallelograms with the given dimensions but with differing lengths for s_1 and s_2. Maybe the area formula gives us some hope:

$$A = \frac{1}{2}(b_1 + b_2)h\ .$$

Putting in the given data gives us the equation:

$$A = 132 = \frac{1}{2}(13 + b_2)(11)\ ,$$

which we can solve for b_2. We multiply both sides by 2 and divide both sides by 11 to get $22 = 13 + b_1$, to find $b_2 = 9$.

Example 13.

The ratio of the length to width of a rectangular photograph is 2 : 5. The longer side is 15 inches.

a. What is the length of the longer side?

b. If the area of the photo is quadrupled, what will the new dimensions of the photo be?

Solution.

a. shorter side is 6 in.

b. new dimensions 12 x 30 in. To visualize, we use grid paper to make a 6x15 rectangle, and find the area which results in 90 units2. We then quadruple it (360 un2.) Thus we need 4 rectangles. We draw a scaled version by making a rectangle of 12x30, showing the length and width both doubled. See Figure 4.

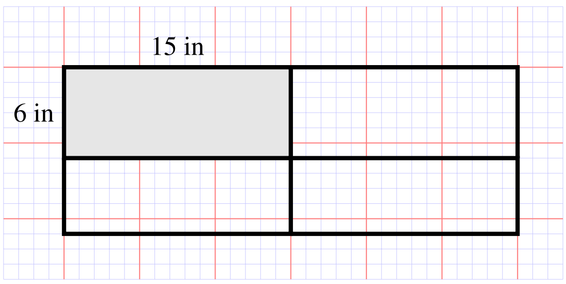


Figure 4

Example 14.

Find the measure of $\angle C$ and $\angle B$ in Figure 5.

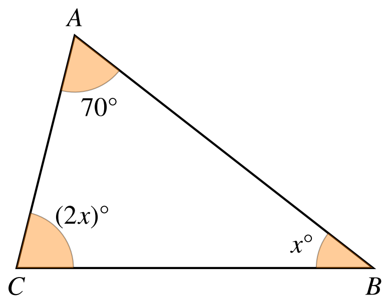


Figure 5

Solution. We want to stress that the figure is a representation to help us think about the problem, it is not necessarily to scale. Additionally we stress that "x" is an unknown, it can take on any value when we write the expression $x + 2x + 66$. Once we set it equal to something, we can find a value for x to make the equation true. This is the very beginning of thinking about "x" as a variable rather than an unknown.

$$
\begin{aligned}
x + 2x + 66 &= 180 \\
3x + 66 &= 180 \quad \text{(Combine like terms)} \\
3x &= 114 \quad \text{(Add the additive inverse of 66 to both sides)} \\
x = \frac{114}{3} &= 38 \quad \text{(Multiply the multiplicative inverse of 3 to both sides)} .
\end{aligned}
$$

Therefore the measure of $\angle B = 38°$, and so the measure of $\angle C = 2 \cdot 38° = 76\circ$.

Example 15.

Pizzas are sold according to diameter. For example, a 6 inch pizza is a pizza with a diameter of 6 inches. At Francesco's pizzeria, there are two pizzas. Pizza A is a 12 inch, and Pizza B has an area of 4150 in^2. Which pizza is bigger? What is the percent of increase from the smaller pizza to the larger pizza?
Solution. To compare Pizza A to Pizza B, let's determine the area of Pizza given the area of Pizza B is 450 in^2. Applying $A = \pi \cdot r^2$ to Pizza A (remember that the radius is half the diameter): $A = \pi \cdot 6^2 = 36\pi$ or about 113 in^2. Therefore, Pizza B is bigger.

The percent of increase from the smaller pizza, Pizza A to the larger pizza, Pizza B, if denoted by p (expressed as a fraction), leads to the equation

$$113 + 113p = 150$$

with the solution $p = 0.327$. Thus the percent increase is 32.7%.

Section 6.2. Write and Solve Equations from Word Problems

In this section we focus on working in two "different directions;" for example, in some sections we challenge students, given a context, find relationships and solutions, while in other sections, students will be given relationships and asked to write contexts. The goal is to help students understand the structure of context in relationship to algebraic representations.

In Chapter 3 we stressed that, in expressing a proportion, it is necessary to be precise about units. If we are thinking that one cup weighs 8 ounces, we might say that the ratio of volume to weight is 1:8. Proper use of units was seen to be essential to understanding scaled drawings. Likewise, the nature of units is necessary to understanding equations, that is, when we write a statement that two expressions are equal, the units on each side of the equation must be the same. To answer the question, "how many cups of water makes 32 ounces?", we write $x = 32$, where x is the number of cups, with ounces as the unit.

In chapter 3 we stressed that, in expressing a proportion, it is necessary to be precise about units: the ratio of volume of water to weight might be 1:8 or 1:2 -in the first case we should read thus as "one cup to 8 ounces" and in the second, "one quart to 2 pounds." And proper use of units was seen to be essential to understanding scaled drawing. And the nature of units is necessary to understanding equations: when we write is a statement that two things are equal, the units on each side of an equation must be the same. To answer the question, "how many cups of water makes 32 ounces?", we write $8x = 32$, where x is the number of cups but what is being equated is number of ounce.

In fact, concentration on the relevant units in a problem may also help us to write an equation. For example, suppose someone drove from Layton to Salt Lake City in 30 minutes. If they traveled at a constant speed, how fast were they driving? Typically we talk about driving speeds by the units of miles per hour. But in this case, we are just given minutes. But in this case, we are just given minutes, that is, it may be helpful to write 30 minutes as $\frac{1}{2}$ hour. Since we want to know an answer in miles per hour, we see that we are missing some information here.

We need to know how many miles it is from Layton to Salt Lake City. A map shows that it is about 24 miles, so if we let s be the speed in miles per hour, we can write the equation

$$s = \frac{24}{\frac{1}{2}}.$$

The units match on both sides of the equation, so we don't need to write them down. We can perform arithmetic without them: $s = \frac{24}{1} \cdot \frac{2}{1} = 48$. And in conclusion, to report our answer we use the units again. The speed was 48 miles per hour.

Solve word problems leading to equations of the form $px + q = r$ and $p(x + q) = r$, where p, q, and r are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach. 7EE4a.

Example 16.

A youth group is going on a trip to the state fair. The trip costs \$52. Included in that price is \$11 for a concert ticket and the cost of 2 passes, one for the rides and one for the game booths. Each of the passes cost the same price. Write an equation representing the cost of the trip and determine the price of one pass.

Solution. We start by identifying an unknown and use a variable to represent it. Since we want to determine the price of one pass, we will say x is the price of one pass.

Next, we identify what we do know. We know that the cost of the trip is \$52. Both of these steps require careful reading of the problem, and translation of these ideas into mathematical language.

What connects the two pieces of information together? Well, the fact that the \$52 trip covers the cost of an \$11 item and two passes, that is: \$52 is equal to \$11 plus the cost of two passes.

We represent the price of each pass by x, we have $52 = 11 + 2x$, which is an algebraic expression of the statement: \$52 is the price of an \$11 item and 2 passes.

This diagram shown is also a helpful way to arrive at the given equation.

<table>
<tr><td>x</td><td>x</td><td>11</td></tr>
<tr><td colspan="3">52</td></tr>
</table>

To solve this equation we may make algebraic steps to isolate x on one side, as follows

$$\begin{aligned} 2x + 11 &= 52 \\ 2x &= 41 \\ x &= 20.5 \end{aligned}$$

Now that we have solved the equation, we interpret the result in context. The price of one pass must be \$20.50.

Does that result make sense? Sure it does, because 2 passes at that price would be \$41. Plus \$11 for the concert ticket and we've got the total price of \$52. Notice how this reasoning is represented and expressed in algebraic equations.

The above example could also be solved by arithmetic reasoning. We began with \$52, \$11 was spent on the concert ticket, subsequently the two passes together cost \$\$52 - \$11 =\$41\$. Given that the cost of the additional 2 passes is the same price, they are each ? of the remaining balance of \$41.00, or \$20.50. Most importantly, the algebraic method encodes this reasoning.

The student may ask, "why are we learning algebraic methods, when the arithmetic method is so easy?" Algebra is a powerful problem-solving tool. However, in order to come to appreciate the value of algebra, students need to encounter problems that are not easily solvable using arithmetic calculations. If they are only asked to solve problems with algebra that could just as easily be solved with arithmetic calculations or by "guess and check," students will likely not see the point in using algebra.

To help bridge the gap between arithmetic and algebraic solution strategies we should put concerted efforts into helping students make connections between arithmetic and algebra, recognizing that formulating an equation is not an intuitive way for many students to represent a problem. We begin by making connections between pictorial and symbolic representations of unknown values, focusing on developing students' abilities to accurately represent word problems in equations.

Encouraging students to write sentences describing algebraic equations will help them learn to model using algebraic equations. For example, students who can translate $C = 2B$ to"There are twice as many carrots as bananas" are close to translating "there are twice as many carrots as bananas" to $C = 2B$.

Example 17.

Brock ate 16 Girl Scout cookies in 5 days (he wasn't supposed to eat any cookies because they belonged to his sister.) The second day he ate 3 more than the first (he felt pretty bad about that.) The third day he ate half as much as the 1st day (he was able to get better control of himself.) The fourth and fifth days, he ate twice each day what he ate the first day (he really likes Girl Scout cookies.) How many cookies did he eat each day?

Solution. The question illustrates the value of having a systematic way of encoding the information that leads to a straightforward algorithm for solving the problem. For such problems, algebra gives us a technique for discovering that some information is missing.

Some general guidelines for solving word problems are:

- Identify what is unknown and needs to be found. Represent this with a variable. x is the amount of cookies Brock ate on the first day
- Determine whether or not you have enough information to solve the equation. If so, write an equation that expresses the known information in terms of the unknown.

$$16 = x + (x + 3) + \frac{1}{2}x + 2x + 2x$$

- Solve the equation. $x = 2$
- Interpret the solution and ensure that it makes sense.

Brock ate 16 Girl Scout cookies in 5 days (he wasn't supposed to eat any cookies because they belonged to his sister.) The second day he ate 3 more than the first; that means he ate 5 cookies on the 2nd day. The third day he ate half as much as the 1st day; eating only 1 cookie. The fourth and fifth days, he ate twice each day what he ate the first day; eating 4 on both days.

$$16 = 2 + 5 + 1 + 4 + 4 \,.$$

Section 6.3. Solve and Graph Inequalities, Interpret Inequality Solutions

Grade 6 content included writing inequalities and graphing them on a number line. For example:

- write an inequality of the form $x > c$ or $x < c$ to represent a constraint or condition in a real-world or mathematical problem;
- recognize that inequalities of the form $x > c$ or $x < c$ have infinitely many solutions;
- represent solutions of such inequalities on number line diagrams.

Grade 7 develops further with solving and graphing one-step and multi-step inequalities using knowledge of solving one-step and multi-step equations. It is important that students understand the similarities and differences between finding the solution to an equation and finding solution(s) to an inequality, and the relationship of each to the real line.

Language is particularly difficult for some students in this section. Phrases like *less than* or *greater than* in the previous section indicated an operation (e.g. subtract or add), in this section they are more likely to suggest <or >. Therefore, making sense of problem situations is critical with writing equations and/or expressions. Central is the ability to predict the type of answers as a way of interpreting how to write the context in algebraic form.

We begin by reviewing the symbols representing inequalities, and explore the work in simplifying inequalities between expressions.

Recall, from the discussion on equivalence at the beginning of this chapter: for a and b two numbers

- $a < b$ means "a is less than b," as in $\frac{1}{2} < \frac{3}{4}$.
- $a \leq b$ means "a is less than or equal to b," as in $\frac{10}{16} \leq \frac{5}{8}$.
- $a > b$ means "a is greater than b," as in $11 > 4$.
- $a \geq b$ means "a is greater than or equal to b," as in $11 \geq \frac{22}{2}$.

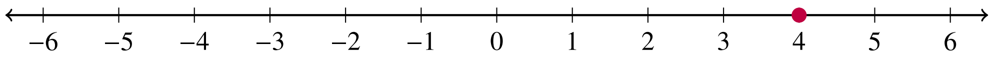
The realization of numbers as points on the line gives another very useful interpretation of these symbols: on the real line $a < b$ means that *a lies to the left of b*, and a ¿ b means that *a lies to to the right of b*.

Consider the statement $x < 4$. There isn't just one value of x that makes this open sentence true. The inequality would be true if $x = 3$ or if $x = 3.9$ or if $x = 0$ or $x = -2$ or $x = \frac{1}{8}$. In fact, it is impossible to list all the values that make this statement true. The solution set for this open sentence contains an infinite number of values. We can resolve this issue by representing solutions of inequalities graphically on the number line.

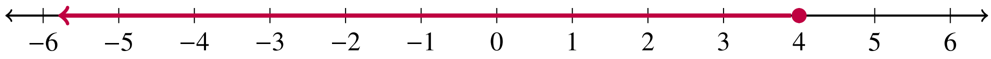
Example 18.

Suppose
$$9x - 10 = 26\ .$$
Use algebra to find the solution, and then represent the solution on a number line.

Solution. Using algebra, we find the solution $x = 4$. To represent a single number on a number line, we draw a filled-in dot on that number, as in the following image:



To show the solution set of the inequality $x \leq 4$ on a number line, we fill in the point $x = 4$ and shade the region on the number line less than 4 like so:



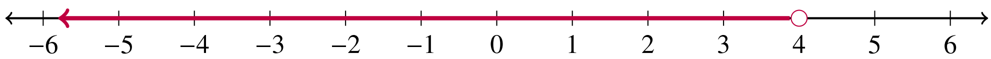
The arrow indicates that the shading should continue forever, so numbers like −4 and even −100 are included.

Now, suppose we are given the relation

$$9x - 10 \leq 26.$$

If we add 10 to both sides of the inequality, then, in the real line representation, everything is shifted to the right by ten units, so we have the relation $9x \leq 36$, which has the same solution set. Now, if we divide by 9, we are just changing the scale by a factor of 1/9, so again we have the same solution set, but now written as $x \leq 4$, with the same graph on the real line as above. Thus this is the graph of the solution set of the inequality $9x - 10 \leq 26$.

Suppose now that we want to indicate $x < 4$ on the number line. We can read this inequality as "x is less than 4" or "x is strictly less than 4" to emphasize that $x = 4$ is not part of the solution set. We represent the solution on the number line as follows:



Notice that the dot over the number 4 is left open to show that the number 4 is not included in the set of values that makes $x < 4$ true.

For the most part we can work with inequalities in much the same way that we work with equations. But there are some important differences too.

As with equations, two inequalities are *equivalent* if they have the same solution set. When we work with equations, we can add (or subtract) the same number to both sides of the equation, and we can multiply (or divide) the same nonzero number to both sides of the equation, and the result is an equivalent equation. Do these same rules apply to inequalities? Let's go through this carefully, using the real line representation of the number system

- Add a number to both sides of an inequality: the solution set does not change. If the number is positive, the effect is to shift everything to the right by that number - by "everything" we mean both sides of the inequality. So if $a < b$, the $a + 10 < b + 10$, and $a - 3 < b - 3$, and so, if E and F are expressions, the solution set of $E < F$ is the same as the solution set $E + 10 < F + 10$ and $E - 3 < F - 3$, and so forth for any number replacing 10 and 3.

- Multiply both sides of an inequality by a positive number a: the solution set does not change. We can think of this as a rescaling: when we replace every number x by ax we are just replacing the unit length 1 by the unit length a. So the relation between two numbers remains the same, and so the solution set of $E < F$ is the same as the solutions set of $aE < aF$.

- Multiply both sides of an inequality by −1 and reverse the inequality: the solution set does not change. Recall that multiplication by −1 takes any point on the line to the point on the other side of 0 of the same distance from 0. So, 7 is two units further away from 0 as is 5, so the same must be true of −7 and −5. But whereas 7 is 2 units to the right of 5, −7 (being two units further away from 0) must be 2 units to the left of 5. Consider the statement: $-x > 3$. This decries the set of all points whose opposite is greater than 3. Now if $-x$ is to the right of 3, then x is to the left of −3; that is $x < -3$. So, we see , using the number line, that multiplication by -1 changes *left of* to *right of* and *right of* to *left of*. $5 < 7$ says that 7 is two units to the right of 5, and $-7 < -5$ says that -7 is two units to the left of five. In general, if $a < b$, then $-a > -b$; that is, for any a and b, both statements are true at the same time, or false at the same time. So, this is true of expressions: $E < F$ and $-E > -F$ have the same solutions set: if a number is inserted for the unknown in these expressions, the two statements are either both true or both false.

Consider for example the statement

$$15 > 10$$

If we multiply both sides by -2, we get $-2 \times 15 = -30$ on the left-hand side and $-2times10 = -20$ on the right-hand side. If we think of the number line or temperature we can see that -20 is to the right of, or is 10 degrees higher than, -30 degrees. So, to get a true statement we have to reverse the "greater than" sign resulting in

$$-30 < -20\,.$$

Solve word problems leading to inequalities of the form $px + q > r$ or $px + q < r$, where p, q, and r are specific rational numbers. Graph the solution set of the inequality and interpret it in the context of the problem. 7EE.4b.

Example 19.

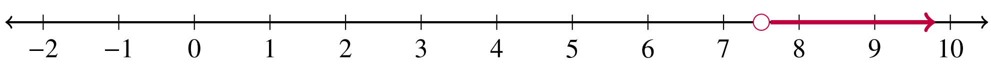
Solve and graph the solution set of the following inequality:

$$8(10 - x) < 20$$

Solution.

$$\begin{aligned}
(10 - x) &< \frac{20}{8} \\
10 - x &< \frac{5}{2} \\
10 - x + (-10) &< \frac{5}{2} + (-10) \\
-x &< \frac{5}{2} - \frac{20}{2} \\
-x &< -\frac{15}{2} \\
x &> \frac{15}{2}
\end{aligned}$$

And the solution set looks like:



Example 20.

Andy has \$550 in a savings account at the beginning of the summer. He wants to have at least \$200 in the account by the end of the summer. He withdraws \$25 each week for food, clothes, and movie tickets. How many weeks will his money last?

Solution. . Recall there is not just one way to get to the solution of an equation, that is, many routes are possible provided each algebraic step is justifiable. The same can be said of inequalities.

Let's consider our known information; Andy starts with \$550, takes away \$25 each week, wants at least \$200 in the end.

Let w = number of weeks the money can last. Our relationship describes the money Andy has, which needs to be more than (greater) or equal to \$200.

$$\begin{aligned} 550 - 25w &\geq 200 \\ -25w &\geq -350 \\ w &\leq 14 \,. \end{aligned}$$

Andy's money will last at most, 14 weeks.

Example 21.

As a salesperson, you are paid \$50 per week plus \$3 per sale. This week you want your pay to be at least \$100. Write an inequality for the number of sales you need to make, and describe the solutions.

Let x be the number of sales made in the week. This week you want,

$$50 + 3x \geq 100.$$

Solving this,

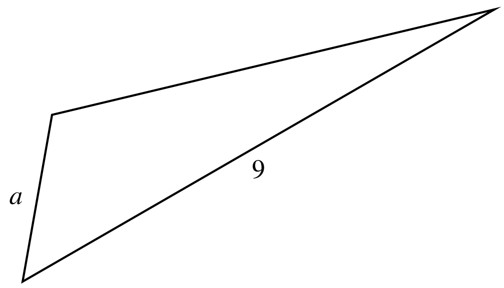
$$\begin{aligned} 50 + 3x &\geq 100 \\ 3x &\geq 50 \\ x &\geq \frac{50}{3} = 16\frac{2}{3} \end{aligned}$$

So, the conclusion is you will need to make at least $16\frac{2}{3}$ sales this week, and since $\frac{2}{3}$ of a sale isn't possible, you will need to make at least 17 sales to exceed \$100 of earnings.

Example 22.

A triangle has a perimeter of 20 units. One side is 9 units long. What is the length of the shortest side?

Let a be the variable to represent the shortest side. Here is a picture.



We know that $a > 0$ because it is a length. Also it is true that $a < 9$ since a is the shortest side. We will see later, that this is not so important.

If the perimeter of the triangle is 20 units, then we must have $a + 9 + b = 20$. So the third side b must be $20 - 9 - a$ units, or $11 - a$ units.

Since a is the shortest side, it must be less than the third side, $11 - a$. So,

$$\begin{aligned} a &< 11 - a \\ a + a &< 11 - a + a \\ 2a &< 11 \\ a &< \frac{11}{2} \end{aligned}$$

If a is less than $\frac{11}{2} = 5.5$, then it automatically is also less than 9 units. So this condition that $a < \frac{11}{2}$ is a stronger restriction than $a < 9$.

In conclusion, $a > 0$ and $a < \frac{11}{2}$. We can write this as $0 < a < \frac{11}{2}$, and it means the value of a lies between 0 and $\frac{11}{2}$.

Summary

To solve a given equation, we perform a sequence of steps that will replace the equation with an equivalent equation, such that the unknown is isolated on one side of the equation. The following actions result in equivalent equations and may be performed to solve an equation:

- Add/subtract the same quantity on both sides of the equation.
- Multiply/divide the same *nonzero* quantity on both sides of the equation.
- Replace an expression or quantity in an equation by an equivalent expression or quantity.

The process of setting up and solving equations is an art. There is not just one way to get to the solution of an equation. Many routes are possible provided each algebraic step is justifiable.

he same process works for solving inequalities. We must remember however, that if we multiply both sides of an inequality by a *negative* number, the direction of the inequality reverses.

Equations and inequalities arise in a variety of contexts in life including financial problems and geometry. If we carefully read the problem and write a corresponding equation or inequality, the properties of numbers and operations can help us do the rest of the work to solve many kinds of problems.

Chapter 7
Probability and Statistics

In this chapter, students develop an understanding of data sampling and making inferences from representations of the sample data, with attention to both measures of central tendency and variability. They will do this by gathering samples, creating plots, representing the data in a variety of ways and by comparing sample data sets, building on the familiarity with the basic statistics of data sampling developed over previous years. They find probabilities, including those for compound events, using organized lists, tables, and tree diagrams to display and analyze compound events to determine their probabilities. Activities are designed to help students move from experiences to general conjectures about probability and number. They compare graphic representations of data from different populations to make comparisons of center and spread of the populations, through both calculations and observation.

Statistics begins in the middle of the nineteenth century during the Crimean War, when Florence Nightingale (a nurse with the British Army) began to notice the excess of British soldiers dying in the hospital of "complications from their injuries" but not from the injuries themselves. Nurse Nightingale suspected that this could be attributed to the sterile (lack thereof) conditions of the field hospitals, and not to the nature of the injuries themselves. She began to gather data, both from field hospitals that acceded to her requirements for sanitary conditions, and those that did not. Her goal was not just to uncover causes, but to suggest remedies that can be implemented instantly. She studied the data, correlating hospital practices with patient mortality, and concluded that there was one rule that, if applied faithfully, would significantly change the result: physicians should wash their hands. When she presented this suggestion to the high command, the response did not meet the urgency she felt. So, she appealed to Queen Victoria: she invented bar graphs to present these data so that the Queen could visualize the significance of her suggestion. It worked: the Queen issued an order that physicians should wash their hands, and that was the beginning of scientific statistics and modern medical practice.

Probability and Statistics are intricately entwined, but historically, the origins are quite distinct. Probability questions arise naturally in games of chance, and over the centuries gamblers placed their faith (and money) on rules, with or without any foundation, had become folklore. In the mid-seventeenth century, the Chevalier de Méré asked the mathematician Blaise Pascal about a rule in a game of dice that, unfortunately, did not work for him. Pascal began a correspondence with Pierre Fermat (two of the leading mathematicians of the century), and between them a theory of probabilities developed that accounted for de Méré's misfortune.Today that theory is fundamental in the study of many processes, especially biological ones, where there is the possibility of random influences on the sequence of events.

Section 1 begins with an exploration of basic probability and notation, using objects such as dice and cards. Students will develop modeling strategies to make sense of different contexts and then move to generalizations. In order to perform the necessary probability calculations, students work with fraction and decimal equivalents. These exercises should strengthen students' abilities with rational number operations. Some probabilities are not known, but can be estimated by repeating a trial many times, thus estimating the probability from a large number of trials. This is known as the Law of Large Numbers, and will be explored by tossing a Hershey's Kiss many times and calculating the proportion of times the Kiss lands on its base.

Section 2 investigates the basics of gathering samples randomly in order to learn about characteristics of populations, in other words, the basics of inferential statistics. Typically, population values are not knowable because most populations are too large and their characteristics too difficult to measure. "Inferential statistics" means that samples from the population are collected, and then analyzed in order to make judgments about the population. The key to obtaining samples that represent the population is to select samples randomly. This is not always easy to do, and an important part of this chapter is to think about what "random sample" means. Students will gather samples from real and pretend populations, plot the data, perform calculations on the sample results, and then use the information from the samples to make decisions about characteristics of the population.

Section 3 uses inferential statistics to compare two or more populations. In this section, students use data from existing samples and also gather their own data. They compare plots from the different populations, and then make comparisons of center and spread of the populations, through both calculations and visual comparisons.

This unit introduces the importance of fairness in random sampling, and of using samples to draw inferences about populations. Some of the statistical tools used in Grade 6 will be practiced and expanded upon as students continue to work with measures of center and spread to make comparisons between populations. Students will investigate chance processes as they develop, use, and evaluate probability models. Compound events will be explored through simulation, and by multiple representations such as tables, lists, and tree diagrams.

The eighth grade statistics curriculum will focus on scatter plots of bivariate measurement data. Bivariate data are also explored in Secondary Math I. However, statistics standards in Secondary Math I, II, and III return to exploration of center and spread, random probability calculations, sampling and inference.

The student workbook begins with the anchor problem: the game "Teacher always wins!" The purpose of this particular activity is to start thinking about what kind of data are needed to resolve a problem (in this case, the apparent unfairness of the game), and secondarily, to illustrate that such resolution is not always as simple as it originally seems. This point is made again towards the end of the first section with the Monty Hall problem. In the meantime, the problem introduces the student to all of the fundamental ideas of this chapter. In order to illustrate these goals, as will be done in the succeeding sections, let's first look at this type of problem in the context of simple games.

What is a fair game?

A *game*, actually, a "simple game", has *players* (2 or more), a *tableau*: the field on which the game is played , *moves*: the set of actions that the players can make on the tableau and finally *outcomes*: the set of end positions on the tableau. Finally there is a rule to decide who is the winner. This may be described as a rule that assigns to each outcome, one of the players declared as winner. We can say this another way: the set of outcomes is partitioned into components, one for each player. When the game reaches an outcome, the winner is the one assigned to the component containing that outcome. Now, for some games there are outcomes that lie in more than one component; in which case the result is called a "tie." A game is called *fair* if all the outcomes are equally likely, and all the components have the same number of outcomes.

a. First game: Two spinner game. This is a game for two players: each has a spinner partitioned into five sectors of equal areas, and each sector has one of the numbers $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ in it, and there is no number common to both spinners. A move consists of a spin by both players, and the winner is the spinner with the higher number.

Is this a fair game? We see right away that it need not be: if player A has $\{0, 1, 2, 3, 4\}$ and player B has $\{5, 6, 7, 8, 9\}$, then player B always wins. So, is there a configuration that is fair? For example, if player A has all the odd digits, and B has all the even ones, is this fair?

To answer such questions we first list all the possible outcomes, and then divide that set into two pieces: *A* where player A wins, and *B* where player B wins. If these sets have the same number of outcomes, then it is a fair game. Now, an outcome of this game is a pair of numbers (a, b), where the A needle lands on the sector marked a, and the B needle lands on b. If $a > b$, then (a, b) goes in the set A; otherwise $a < b$ and (a, b) goes in the set B.

Example 1.

Suppose that player A's spinner has the odd digits and B's spinner has all the even digits. Is this a fair game? Is there any configuration that gives rise to a fair game?

Solution. The sample space of outcomes consists of all pairs of numbers (a, b) where a is an odd digit, and b is an even digit. Since there are 5 even digits and 5 odd digits, the number of pairs (a, b), with a odd and b even is 25. Since 25 is odd, we cannot split the sample space into two sets, both with the same number of outcomes. So this cannot be a fair game. But which player has the edge, A or B? The answer to the second question is "no" by the same reasoning, although sometimes A may have the edge, and sometimes B.

Example 2.

Suppose that A's spinner has 6 sectors, marked {0, 2, 4, 6, 8, 10} and B's spinner has the 5 sectors {1, 3, 5, 7, 9}. Is this a fair game?

Solution. This time there are $6 \times 5 = 30$ possible outcomes, so this could be a fair game. The event "A wins" consists of all pairs (a, b) with $a > b$. If $a = 0$, A loses to all of B's spins; if $a = 2$, A loses to 4 of B's spins, and if $a = 4$, A loses to 3 of B's spins, and so forth. So, in all A loses in $5 + 4 + 3 + 2 + 1 = 15$ outcomes. Since there are 30 outcomes, A also wins in 15 outcomes, so this is a fair game

b. Second game: Player B always wins. In this game the tableau consists of four spinners, Red, Blue, Green, Yellow, each with three sectors marked with these numbers:

$$\text{Red} : \{3, 3, 3\} \quad \text{Blue} : \{4, 4, 2\} \quad \text{Green} : \{5, 5, 1\} \quad \text{Yellow} : \{6, 2, 2\}$$

First, player A selects a spinner and then player B selects a spinner from among those remaining. Now, they spin the spinners, and the player who shows the higher number wins. In case of a tie, they spin again. . Let's analyze one set of choices: suppose A picks Blue, $\{4, 4, 2\}$, and B picks Yellow, $\{6, 2, 2\}$. There are nine outcomes, that is all pairs (a, b) where $a$is the number spun by player A, and b is a the number spun by player B. Since there are repetitions, let us distinguish the pairs by their places, so A has $\{4_1.4_2, 2\}$, and B has $\{6, 2_1.2_2\}$. Now we can count the wins:

$$\text{A wins :} \ (4_1, 2_1),\ (4_1, 2_2),\ (4_2, 2_1),\ (4_2, 2_2)\ ;$$

$$\text{B wins :} \ (4_1, 6),\ (, 4_2, 6),\ (2, 6)\ ;$$

$$\text{Tie :} \ (2, 2_1), (2, 2_2)\ .$$

Since a tie leads to the same scenario, where A has more wins than B, no number of ties will compensate for the fact that the odds favor an A win. So a this is not a fair game.

Example 3.

Once A has made a choice, is there a particular choice for B that favors B winning? Hint: we wouldn't ask the question if the answer weren't "yes."

Solution. Another clue is the label of this game. In fact, the odds favor player B if B always chooses the spinner listed directly after the spinner chosen by A! (This means, for example, that if A picks the yellow spinner, B should pick the red).

c. Third game: Four spinning players. Now let's have four players: Red, Blue, Green and Yellow, one for each spinner. Each player spins, and the highest number wins.

Example 4.

Is this a fair game?

Solution. There are $3 \times 3 \times 3 \times 3 = 81$ outcomes, since each of the four spinners can produce three numbers. For this to be a fair game, we would have to partition these 81 outcomes into four events, all with the same number of outcomes. Since $81/4$ is not an integer this cannot be done. Wait! Maybe there is a tie. Well, there are no ties, for the only duplicated number is 2, and if Blue and Yellow show a 2, Red always shows a 3, so Red wins if Green shows a 1, otherwise Green wins. To see if this is a fair game: count the number of outcomes that produce a win for each player.

The result is this: Red wins in 6 outcomes, Blue in 12, Green in 36 and Yellow in 27. At first it seems surprising that when only two spinners are used, we can have a bias toward any color, but when all four are spun, Green wins by a long shot.

In the above discussion we used the term *odds*.This term is used to describe the ratio among a set of events that partition a sample space. This is not an easy phrase to digest, so let us illustrate. In flipping a fair coin, the odds of "heads" to "tails" are 1:1. In rolling a fair die, the odds of the numbers turning up is 1:1:1:1:1:1. However, the odds of getting a number divisible by 3 are 2:4 since the set {1,2,3,4,5,6} has two numbers divisible by 3, and four that are not. We could also say that the odds are 1:3, since these both describe the same ratio.

In example 1 the odds of A winning are 15:10, or 3:2; in example 2, the odds vary, depending upon the choice A makes, but (if B makes the right choice), the odds always favor B (that is the odds are $a : b$, with b always larger than a. In example 3, the odds are 6:12:36:27, or (since all numbers are divisible by 3), 2:4:12:9.

The message here is simply, uncovering biases in a game is not easy; one must choose outcomes and assign them to players as wins according to the given rules.

Section 7.1: Analyze Real Data and Make Predictions using Probability Models.

The importance of an understanding of probability and the related area of statistics to becoming an informed citizen is widely recognized. Probability is rich in interesting problems and provides opportunities for using fractions, decimals, ratios, and percent. For example, when we ask "what are the chances it will snow today?" or "what are the chances I will pass my math test?" or any question filled with the phrase "what are the chances?" we are really asking "what is the probability that something will happen." The subject of probability arose in the eighteenth century in connection with games of chance. But today it is an essential mathematical tool in much of science; in any phenomenon for which there is a random element (such as life), probability theory naturally comes up. For example, the basic concept in the life insurance business is to understand, in any given demographic, the probability of a person of age X living another Y years. In business and finance, probability is used to determine how best to allocate assets or premiums on insurance. In medicine, probability is used determine how likely it is that a person actually has a certain disease, given the outcomes of test results.

This section starts with a review of concepts from Chapter 1 Section 1 and then extends to a more thorough look at probability models. A complete probability model includes a sample space that lists all possible outcomes, together with the probability of each outcome. The probability may be considered as the relative frequency of the model (that is, in an ideal situation, the fraction of times the given outcome comes up in a given number of trials. The sum of the probabilities from the model is always 1 (reflecting the fact that any experiment always leads to an outcome in our sample space). A *uniform probability model* will have relative frequency probabilities that are the same, namely $1/n$ where n is the number of distinct outcomes. An event is any set of possible outcomes. A probability model of a chance event (which may or may not be uniform) can be approximated through the collection of data and observing the long-run relative frequencies to predict the approximate relative frequencies. Probability models can be used for predictions and determining likely or unlikely events. There are multiple representations of how probability models can be displayed. These include, but are not limited to: organized lists, including a list that uses set notation, tables, and tree diagrams.

Approximate the probability of a chance event by collecting data on the chance process that produces it and observing its long-run relative frequency, and predict the approximate relative frequency given the probability. For example, when rolling a number cube 600 times, predict that a 3 or 6 would be rolled roughly 200 times, but probably not exactly 200 times. 7.SP.6.

In Chapter 1 we discussed the experiments of John Kerrich with coin tosses. If we wanted to reproduce his experiment, we'd have to accept that it takes a considerable amount of time to toss a coin 10,000 times as he did. We might want to look for an easier way of reproducing the same situation and run that. This is what is called a *simulation*: a parallel experiment that has all the same properties as the coin toss. Such, for example, is possible using a spreadsheet on a computer. Spreadsheets (such as Excel) have a *random number generator*; it could be arranged to produce a 0 or a 1 (or an H or a T) at random. The spreadsheet can be programed to do this 10,000 times in just a few seconds, and tabulate the results. We could also simulate the spinner games by programming the random number generator to produce of on the first six whole numbers at random.

At this point, it will help to think of a *random process* as an experiment (to be repeated many times) whose outcome is one of a well-defined set of outcomes, all of which are equally likely. Actually, this is a *uniform random process*; we may have some reason to assign different probabilities to each outcome; this would be a *non-uniform* random process. Repeated tosses of a fair coin, repeated twirls of a fair spinner, (one in which all sectors have the same central angle), repeated tosses of a fair die; these are all uniform random processes. However, in more complicated situations, it depends upon what is seen as the outcome. For example, suppose the experiment is to toss a pair of dice, and the outcomes are the possible sums. Then the possible set of outcomes is the set of whole numbers between 1 and 12. Clearly, the outcomes are not equally likely: a 1 is impossible, and a 2 is far less likely than a 7, and so forth.

Example 5.

The athlete Maria wants to understand the randomness in her success at making baskets from the free throw line. She knows that she has about 50% accuracy, but that doesn't mean that she scores every other time she shoots the basket. Sometimes she can make two or three in a row; and other times she can miss as often. Maria would like to understand if this is about her, or a natural consequence of the randomness. So, she decides to simulate shooting 25 free throws. Using a coin (or a computer program) she lets heads represent the ball going into the basket and tails represent the ball missing the basket. Each toss of the coin represents a shot at the free throw line.

Maria begins by making a table or filling in the one below. She will mark an '*x*' in the appropriate column for each toss of the coin. Here are her first ten entries:

Toss number	Made the basket (heads)	Missed the basket (tails)
1	x	
2		x
3		x
4	x	
5	x	
6		x
7	x	
8	x	
9	x	
10		x

Questions for Maria to consider:

1. How many free throws went into the basket?

2. How many free throws missed the basket?
3. What outcome does she expect in the next free throw?
4. What is the theoretical probability that the next toss of the coin will show success (that is, come up heads)?
5. Suppose that we replace Maria's estimate of her rate of accuracy being 50% by her actual experience, that in the long run, she makes the basket two times out of three. Can you devise a new model based on this assumption of Maria's ability?

Solution. The answers to the first two questions are 6 and 4 respectively. Questions **3** and **4** are more interesting. Since she knows that, on average, she gets one basket out of every two attempts, her expectation is that she will make the next basket. If instead she misses, her expectation of a "make" on the next try is even stronger, and so forth. However (in answer to **4**), the theoretical probability of a "make" on any attempt is *always* 50%, no matter how many misses precede the attempt. Probability theory tells us that in the long run, she'll have successes to make up for a sequence of misses, but we have no way of knowing on which attempt this will start. As for **5**, one possible simulation is to roll a die: a 1 or a 2 represents a missed basket; otherwise, she scores.

The objective for students in this section is to collect data from an experiment representing a random process. It may be a simulation, as in the above example, or it may be based on actual trials (in the case of the example, Maria actually shooting free throws). The goal is to recognize that as the number of trials increase, the experimental probability approaches the theoretical probability. This tendency is called the *Law of Large Numbers* In this standard we focus on relative frequency: the ratio of successes to the number of trials. The Law of Large Numbers tells us that, as the number of trials increases, this ratio should get closer and closer to the actual (or theoretical) probability. So, in the case of Maria's simulation, those ratios (expressed as decimals are

$$1.0, 0.5, 0.33, 0.5.0.6, 0.5, 0.57, 0.62, 0.66, 0.6\ .$$

We see a tendency toward 0.5, the probability of heads in a coin toss, but there haven't been enough trials to distinguish the result from the theoretical probability of the simulation to Maria's experience that she should make 2 baskets out of 3.

A *fair game* is one in which each player has an equal chance of winning the game. Tossing a coin is considered a fair game, since there is an equal chance that a head or a tail will come up. Maria shooting baskets alternately with the point guard of her school basketball team is probably not a fair game (unless Maria is the point guard). Keep in mind, because a game is fair, this doesn't mean that in any set of repetitions the wins will be equal; one could toss a fair coin six times and get six heads.

Develop a probability model and use it to find probabilities of events. Compare probabilities from a model to observed frequencies; if the agreement is not good, explain possible sources of the discrepancy.

Develop a uniform probability model by assigning equal probability to all outcomes, and use the model to determine probabilities of events. For example, if a student is selected at random from a class, find the probability that Jane will be selected and the probability that a girl will be selected.

Develop a probability model (which may not be uniform) by observing frequencies in data generated from a chance process. For example, find the approximate probability that a spinning penny will land heads up or that a tossed paper cup will land open-end down. Do the outcomes for the spinning penny appear to be equally likely based on the observed frequencies? 7.SP.7.

Example 6. The Addition Game.

Roll two dice (also called *number cubes*) 36 times. On each roll:

if the sum of the two faces showing up is odd, player #1 gets a point;

If that sum is even, player #2 gets a point.

The winner is the one with the most points after 36 rolls. Is This a fair game?

a. Play the game. Based on your data, what is the experimental probability of rolling an odd sum? An even sum?

P(odd) = ...   P(even) = ...

b. Find all the possible sums you can get when rolling two dice. Organize your data.

c. What is the theoretical probability of rolling an odd sum? An even sum?

P(odd) = ...   P(even) = ...

d. Do you think the addition game is a fair game? Explain why or why not.

Let us pause at this point to review the basic concepts of the probability theory as it has been developed so far, continuing from Chapter 1 . First of all, in every example, be it tossing a coin, rolling a die or twirling a spinner, we are concerned with an *experiment*: that is, an activity that can result in one of a set of *outcomes*, and - in our context - we may have defined "success" by a certain subset of outcomes. For a tossed coin, the set of outcomes is $\{H, T\}$; and if our interest is obtaining a head, then the success outcome is H. For a rolled die, the outcomes are the faces: $\{1, 2, 3, 4, 5, 6\}$. If success is defined as getting an even number, then the set of outcomes of interest is $\{2, 4, 6\}$. For any experiment, we need to list all possible outcomes: this list is called the *sample space*. A subset of the sample space is called an *event*. In a given context, a certain subset of the sample space is the event we are aiming for: that is the "success."The experiment is usually run many times in an attempt to discover what the probability is of success. This leads to what we call the *experimental probability*. In contrast, the *theoretical probability* (in a uniform random process) is the quotient of the number of outcomes in the success event divided by the total number of events. Often the set of possible outcomes is so large, that we estimate the probability of success by experiments, or by modeling. For example, instead of doing what John Kerrich did, we can now model 10,000 coin tosses by a con-tossing machine, or by a computer program that randomly chooses H or T successively for many times.

Example 7.

Suppose there are 100 balls in a bag. Some of the balls are red and some are yellow, but we don't know how many of each color ball are in the bag. Bailey reaches into the bag 50 times and picks out a ball, records the color, and puts it back into the bag. Here the sample space is the set of balls in the bag, and the "success" event is the set of yellow balls in the bags. If Bailey picked 16 yellow balls, then the experimental probability of picking a yellow ball is:

$$16/50 = 32\% \ .$$

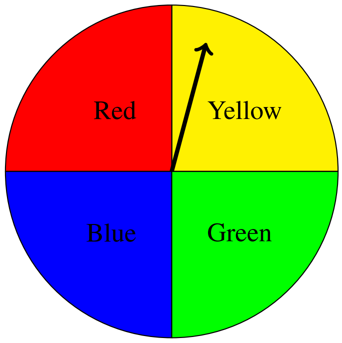
This 32% is likely to be close to the theoretical probability of picking a yellow ball, since 50 is quite a large number, relative to 100. However, if she had only picked 5 balls, three of which were yellow, then her experimental probability would have been 60%. Clearly Bailey will have more confidence in the experimental probability of the larger sample.

Find probabilities of compound events using organized lists, tables, tree diagrams, and simulation. Understand that, just as with simple events, the probability of a compound event is the fraction of outcomes in the sample space for which the compound event occurs.

Develop a probability model (which may not be uniform) by observing frequencies in data generated from a chance process. 7.SP.8.

Example 8.

In some games that use spinners, the spinner is equally likely to land in red, yellow, blue, or green. If Sarah is allowed two spins, what are all the possible outcomes?



Solution. We could make an organized list, table or tree diagram to show all the possible outcomes.

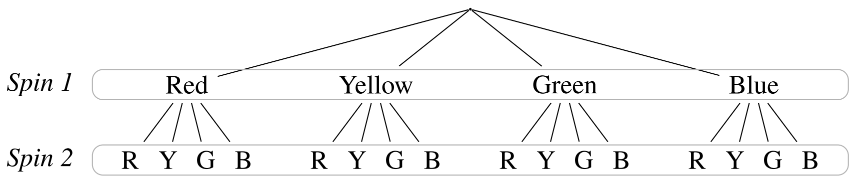
Organized List

Red, Red	Yellow, Yellow	Green, Green	Blue, Blue
Red, Yellow	Yellow, Red	Green, Red	Blue, Red
Red, Green	Yellow, Green	Green, Yellow	Blue, Yellow
Red, Blue	Yellow, Blue	Green, Blue	Blue, Green

Table

		Spin 2 Red	Yellow	Green	Blue
Spin 1	**Red**	RR	RY	RG	RB
	Yellow	YR	YY	YG	YB
	Green	GR	GY	GG	GB
	Blue	BR	BY	BG	BB

Tree Diagram (Vertical)



To determine how many different possible outcomes there are to this two-stage experiment first observe that there are 4 possible outcomes for the first spin (Spin 1) and four possible outcomes for the second spin (Spin 2). Each of the methods show that there are 16 different paths, or outcomes, for spinning the spinner twice. Rather than count all the outcomes, we can actually compute the number of outcomes by making a simple observation. Notice that there are four colors for the first spin and four colors for the second spin. We can say there are 4 groups of 4 possible outcomes which gives 4×4 (or 16) possible outcomes for the two-stage experiment of giving this spinner two spins.

Fundamental Counting Principle

If an event A can occur in m ways and event B can occur in n ways, then events A and B can occur in $m \cdot n$ ways. The Fundamental Counting Principle can be generalized to more than two events occurring in succession.

Example 9.

What is the probability that in 2 spins, the spinner will land first on blue and then on yellow? Table 2 below shows the outcomes both as a fraction F, and a percent P, of spinning the spinner.

	RR	RY	RG	RB	YR	YY	YG	YB	GR	GY	GG	GB	BR	BY	BG	BB
F	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
P	6.25 %	6.25%	6.25%	6.25%	6.25%	6.25%	6.25%	6.25%	6.25%	6.25%	6.25%	6.25%	6.25%	6.25%	6.25%	6.25%

Table 2

Since each of the 16 spin outcomes is equally likely, and spinning a blue and then a yellow is just one event, its probability 1/16=6.25%. Note that the sum of the probabilities of all outcomes is one, indicating that all possible outcomes are shown.

We may also ask: what is the probability of spinning a blue and a yellow, in either order? Looking at the list, this event has two possible outcomes : YB and BY, so its probability is (1/16) +(1/16) = 1/8.

Example 10.

Roll two dice and examine the top faces.

a. What is the probability of rolling two dice and getting two threes?

b. What is the probability of getting any pair?

c. What is the probability of at least one die showing a three?

d. What is the probability of getting a three and an even number?

Solution.

a. Here the sample space is all possible outcomes of rolling two dice: that is, all pairs (a, b) where a and b each run through the integers 1,2,3,4,5,6. There are 36 such pairs, and only one is (3,3). Thus the theoretical probability is 1/36.

b. Since there are 6 doubles, the probability is 6/36, or one-sixth.

c. We look at all pairs $(3, b)$: there are six of them. Now look at all pairs $(a, 3)$: here are another six. However, the pair (3,3) has been counted twice, so the number of pairs with at least one three is 6+6-1 = 11, and the probability of rolling at least one three is 11/36. Students may want to perform this experiment 36 times to see if they get an experimental probability close to this theoretical probability.

d. From **c** we know that there are 11 outcomes with at least one three. From these listed outcomes, the ones that show a 3 on one face and an even on the other are $\{(3, 2), (3, 4), (3, 6), (2, 3), (4, 3), (6, 3)\}$. Thus there are six outcomes in the success event: a three and an even number. The theoretical probability then is 6/36, or 1/6.

There is another way of doing part **a**. The proposed experiment is the same as that of rolling one die twice in a row. In order to get two threes we must get a three on the first roll, and then a three on the second. The chance of getting a three in the first roll is 1 in 6 (and thus a probability of 1/6). After that there is again a 1 in 6 chance of getting a three on the second roll. Therefore, the chance of getting a three on two consecutive rolls is 1/6 of 1/6 or $\frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$. We can now also answer the question: what is the chance of rolling three consecutive threes? Well, there is a 1 in 36 chance of getting two consecutive threes, and after that 1 in 36, a 1 in 6 chance of getting a third three. So altogether there is a (1/36)(1/6) = 1/216 chance of getting three consecutive threes.

An event of this type is called a *compound event*; that is, a compound event is an event that can be viewed as two (or more) simpler events happening simultaneously. If the simple events do not influence each other, they are called *independent events*. When rolling a die two times in a row, the two roles are independent. In this case, we can calculate the probability of the compound event as the product of the probabilities of the simple events. For example: what is the probability of rolling a three and then an even number? We analyze the problem this way: the probability of rolling a three is 1/6, and that of rolling an even number 1/2. Thus the probability of rolling a 3 *and then* an even number is (1/6)(1/2) = 1/12.

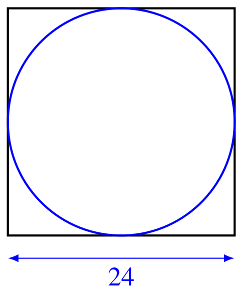
Example 11.

On average, Maria scores 10 or more points in 50% of the games, and Izumi does so in 40% of the games. What is the probability of both Maria and Izumi scoring 10 points in a game?

Solution. This is a compound event, so we look at it this way: the (experimental) probability that Maria scores 10 points or more is 0.5, and that for Izumi is 0.4 so the probability that both will happen is the product, (0.5)(0.4) = 0.2. Therefore, in 1/5 of the games, Maria and Izumi will each score 10 or more points.

Example 12.

Jamal is preparing for this competition: a square of side length 24 inches with an inscribed circle (see the diagram) is placed on the floor, and a line is drawn 8 feet away from the square. Competitors have to toss a small bean bag from behind the line, and get a point if the bag lands inside the circle. Jamal has honed his skills so that he knows that he can hit the square every time, but otherwise cannot affect where it lands. Given this he wants to determine the probability that he will strike the target somewhere within the circle.



Solution. Here the sample space is the square; due to Jamal's skill. Success is defined as landing in the circle, so the probability of success is the quotient of the area of the circle by the area of the square: the square has area 48 in $\times$ 48 in = 2304 in^2. The area of the circle is $\pi \cdot r^2 = \pi(24^2)$ or, approximately, 1809.56 in^2

The probability that Jamal would strike anywhere within the circle's target range would be

$$\frac{1809.56}{2304} = 0.785 = 78.5\% \ .$$

.

Now we move on to more complicated examples in order to demonstrate the value of tables, organized lists and tree diagrams in order to determine probabilities in compound experiments.

Example 13.

Ted and Mikayo are going to play a game with one die. At each toss Ted wins if the upturned face is even, and Mikayo wins if that face is odd.

Solution. The sample space consists of the set of all possible outcomes 1, 2, 3, 4, 5, 6. The event

Even is the set of outcomes 2, 4, 6 and Odd is the set 1, 3, 5. Assuming a fair die, that is, all outcomes are equally probable, then since the sample space is partitioned evenly into the events "Ted Wins" and "Mikayo Wins," this is a fair game.

Example 14.

After a while, Ted and Mikayo get bored, and change the game. The die is rolled twice, and Ted wins if the sum is even, and Mikayo wins if the sum is odd. The sample space is now the set of all outcomes of two rolls of the die, that is (a, b), where a and b run through all positive integers less than or equal to 6. The event making Ted the winner is "a + b is even." and the event Mikayo wins by is "a + b is odd." Check that this is a fair game.

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Example 15.

Now Ted and Mikayo turn to spinner games. They each take a spin, and Ted is the winner if there is at least one green; otherwise Mikayo wins. In Table 1, all the outcomes are equally possible. The event that Ted wins: "at least one green' has 7 outcomes, so the probability that Ted wins is 7/16, and that Mikayo wins is 9/16. Note that this is not a fair game!

So Ted suggests a new game: Ted wins if there is at least one green or one yellow, and Mikayo wins if there is at least one red or one blue in three spins. Sounds fair, but is it? Explain.

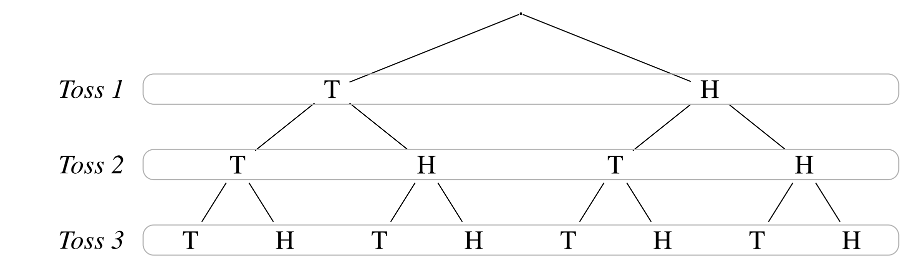
Example 16.

Kody tosses a fair quarter three times. What is the probability that two tails and one head in any order will result?

Solution. Tossing a coin repeatedly involves independent events. For example, the outcome of the first coin toss does not affect the probability of getting tails on the second toss. Kody's list of the sample space for the toss of three coins is written as

$$\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

and can be displayed as a tree diagram, with accompanying outcome and probability.



Outcome	TTT	TTH	THT	THH	HTT	HTH	HHT	HHH
Probability	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

This tree diagram makes clear that there are eight possible outcomes for the experiment "toss a coin three times," all of which are equally likely, so each has probability 1/8. We could also consider this a compound event, made up of the three successive events "toss a coin." Since the probability of each outcome of each event is 1/2, we multiply the probabilities, and again find that each outcome of three tosses has $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$ probability.

Now we can solve the problem. The event of two tails and one head in any order consists of the three outcomes: HTT, THT, and TTH, so the probability of this event is 3/8 The event "at least one tail" consists of all outcomes but for HHH, thus has seven outcomes, and its probability 7/8.

This example illustrates certain important principles for finding probabilities. First of all, given an experiment to be perfumed, we first determine the set of all possible outcomes. The outcomes must be mutually exclusive, that is, it cannot happen that two outcomes can happen simultaneously. So, in rolling a pair of dice, we cannot have one outcome be "the sum of the dice is 7", and another be, "one of the dice is a three," because the roll (3,4) is in both of these outcomes. It is important that the outcomes be most elementary observations that could be made. In the case of rolling a pair of dice, the outcomes are pairs of integers between 1 and 6. Then "the sum of the dice is 7" is the event consisting of all pairs, the sum of whose faces is 7. With this understanding, the probability of an event is the sum of the probabilities of the outcomes in that event.

To illustrate with the example of the roll of three dice, the event is "2 tails and 1 head", and it consists of the three outcomes HTT, THT, TTH, so has probability 3/8.

More can be said: if have two events in a given experiment that have no outcomes in common, then the probability of either event happening is the sum of the probabilities of the two events. Consider, for example, the probability of getting either precisely two heads or precisely two tails in three coin tosses. Since there are only three coins, we cannot have both two heads and two tails, so there is no outcome common to both. Thus the probability of either precisely two heads or precisely two tails is 3/8+3/8 = 3/4.

When do we add probabilities and when do we multiply them? If an event can be viewed as either of two events *in the same experiment* happening, and the two two events have no outcomes in common then we add their probabilities to get the probability of the main event. If an event can be viewed as two events in *different experiments* happening simultaneously, we multiply the probabilities of the component events to find the probability of the main event. This is analogous to working with lengths: when we add lengths we get another length, but when we multiply lengths we get area.

Section 7.2: Use random sampling to draw inferences about a population

In this section students will be looking at data of samples of a given population, and then making inferences from the samples to the population. Students will utilize graphs of data along with measures of center and spread to make comparisons between samples and to make an informal judgment about the variability of the samples. After examining the samples, then students make conclusions about the population.

It is important that students think about the randomness of a sample as well as how variations may be distributed within a population. These ideas are quite sophisticated. Activities within this section are designed to surface various ideas about sampling.

Understand that statistics can be used to gain information about a population by examining a sample of the population; generalizations about a population from a sample are valid only if the sample is representative of that population. Understand that random sampling tends to produce representative samples and support valid inferences.7.SP.1

EXAMPLE 17.

Mrs. Moulton was curious to know what proportion of the 7th grade students at her school chose pizza as their favorite menu item for lunch from the school cafeteria. She asked her 7th grade 1st and 2nd periods to help her find an answer to this question. Mrs. Moulton's students realized that they will not be able to interview every 7th grader so instead, they considered the 1st and 2nd period classes to be 'random samples' and took a poll in each class to determine if their favorite lunch menu item was pizza, hamburger, salad, or if the question didn't apply, because either they brought a lunch from home or had no lunch at all. Once the data were gathered, the students were asked to answer the following questions:

a. Create a bar graph to view the sample data for each class, using percentage data.

b. Describe the differences and similarities between the data from the two classes.

c. Based on these two samplings, do you think your class data is representative of the 7th grade?

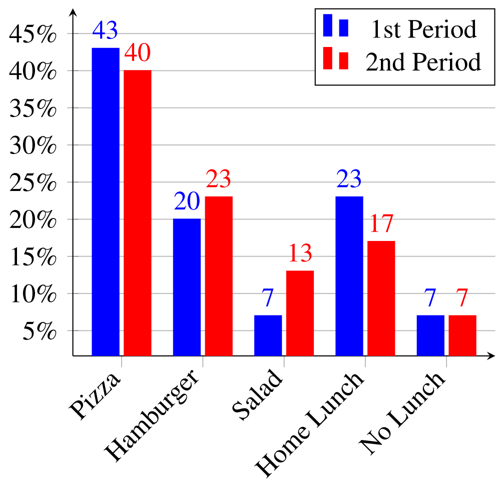
d. If the sampling is representative, what conclusions could you draw?

e. Create a bar graph of the combined sampling, using percentage data.

f. Compare your original class sample and the combined sampling, using percentage data.

SOLUTION.

a. This is Graph 1.



When comparing data from two populations, it is more useful to use percentage data (called a *relative frequency graph*) rather than the raw data. For example, if Mrs. Moulton's first class had 40 students, of whom 15 selected "pizza" and her second class had 60 students of whom 20 selected "pizza", then the raw data graph would suggest that pizza were more highly favored in the second class than in the first. But in fact, only 33% of the second class chose pizza, while over 37% in the first class did so. More importantly, Mrs. Moulton wants to predict the *proportion* of students favoring pizza, and so we should be looking at proportional, not raw, data.

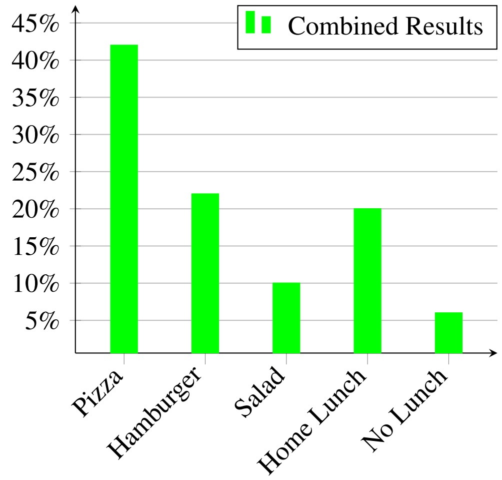
b. Some students may say that the graphs are almost the same, others may say that a greater percentage of students in the second class preferred salad. Others may say that there is no difference, for the ranking order (pizza, hamburger, home lunch, salad, no lunch) is the same for both classes. The question really is "are there *significant* differences,?" and students should know that there are statistical measures of significance, but their use has to be justified in terms of the context. In this case, relative to the question posed by Mrs. Moulton, we would conclude that there is no significant difference.

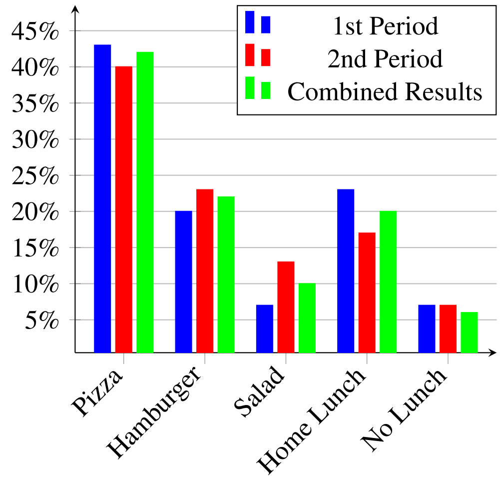
c. Of course this is a subjective question, but students should learn that there are statistical measures of the amount of confidence the researcher can put in these graphs, and again, that use of those measures has to be justified in terms of the context

d. Accepting the sampling as representative, it s fair to conclude that more than 40% of 7th graders choose "pizza."

e. Graph 2 is a relative frequency graph for the combined data, and Graph 3 has put them all together.

f.





There is little doubt that, as far as the question (what proportion of 7th grade students prefer pizza?), the conclusion stated in part **d** is confirmed by either data set and the combined data set. One might go further and express confidence in the ordering of the preferences, and that the "significant" disagreement in the "salad" casts doubt on any prediction of the proportion of salad eaters in the class.

What do we mean by a random sample? A random sample is a subset of individuals (a sample) chosen from a larger set (a population). Each individual is chosen randomly and entirely by chance, such that each individual has the same probability of being chosen at any stage during the sampling process, and each subset of k individuals has the same probability of being chosen for the sample as any other subset of k individuals. A simple random sample is an unbiased surveying technique.

In the student workbook homework section 7.2a, students will work on a problem titled Inquiring *Students Want to Know!* The premise of the problem is to make a list of topics of student interest and to design a sampling method for collecting date from 10 or more randomly selected students. In the teacher notes a recommendation is made to have students ask every tenth student who comes into the school. This type of sampling is called systemic sampling.

Systematic sampling is an additional statistical method involving the selection of elements from an ordered sampling frame. The most common form of systematic sampling is an equal-probability method. In this approach, progression through the list is treated circularly, with a return to the top once the end of the list is passed. The sampling starts by selecting an element from the list at random and then every kth element in the frame is selected, where k, the sampling interval is calculated as $k = N/n$, where n is the sample size, and N is the population size.

Using this procedure each element in the population has a known and equal probability of selection. This makes systematic sampling functionally similar to random sampling and is typically applied if the given population is logically homogeneous, because systematic sample units are uniformly distributed over the population.

Drawing conclusions from data that are subject to random variation is termed *statistical inference*. Statistical inference or simply 'inference" makes propositions (predictions) about populations, using data drawn from the population of interest via some form of random sampling during a finite period of time. The outcome of statistical inference is typically the answer to the question "What should be done next?"

Random sampling allows results from a sample to be generalized to the population from which the sample was selected. The sample proportion then, is the best estimate, given the constraints of the population proportion. Students should understand that conclusions drawn from random samples can then be generalized to the population from which the sample was appropriately selected. The sample result and the true value from the entire population are likely to be very close but not exactly the same. Understanding variability in the samplings allows students the opportunity to estimate or even measure the differences.

Use data from a random sample to draw inferences about a population with an unknown characteristic of interest. Generate multiple samples (or simulated samples) of the same size to gauge the variation in estimates or predictions. For example, estimate the mean word length in a book by randomly sampling words from the book; predict the winner of a school election based on randomly sampled survey data. Gauge how far off the estimate or prediction might be. 7.SP.2

The primary focus of 7.SP.2 is for students to collect and use multiple samples of data (either generated or simulated), of the same size, to gauge the variations in estimates or predication, and to make generalizations about a population. Issues of variation in the samples should be addressed by gauging how far off the estimate or predication might be.

Was Mrs. Mouton's technique for gathering data effective? To check on this, and to see how far off the estimate might be, Mrs. Moulton visits the school district's food services website and learns the actual percentages of 7th graders' consumption over the academic year: 40% favor pizza, 20% favor hamburgers, 10% favor salad, 20% bring a home lunch, and 10% are unaccounted for (which is calculated as having no lunch). This is pretty strong confirmation. Another approach would be to ask each student in each grade to pick a student at lunch at random and check what the choice was. This might work, but because of the group of students who bring their own lunch or have no lunch, it could be skewed. In summary, selection of random samples, and representation of the data by graphs are often questions of taste or common consent,

We often use bar graphs for comparing categorical data and making inferences about populations from the graphs. If comparisons are being made between unequal sized groups, percents give a better basis of comparison. However if the samples are equal in size, then either counts or percents give comparable graphs. For categorical data such as this, either bar graphs or pie charts are appropriate. If bar graphs are made, they should be called bar graphs, rather than "histograms" because the data are categorical. Histograms are used for graphs that display a range of numerical values, such as heights, or ages (which we will discuss further in Section 7.3). Although histograms can be used in drawing the graph, this may generate graphs that do not truly represent the data, unless each histogram is equal in size.

The variability in the samples can be studied by means of simulation. Although we have used this word before, in a colloquial sense, it is now desirable to work on making the concept somewhat more precise A *simulation* is an experiment that models a real-life situation and helps students develop correct intuitions and predict outcomes analogous to the original problem. Now, Mrs. Moulton wants to create a simulation that models the eating habits of 7th graders. Here is her model: she prepares a large non see-through bag, with 200 red skittles (representing pizza), 100 purple skittles (representing hamburgers), 50 green skittles (representing salad), 100 yellow skittles (representing home lunch) and 50 orange skittles (representing no lunch). These proportions are close to the proportions shown in Graph 2. The purpose is to create a population of 500 (representing the amount of students in the school) with 40% red skittles for pizza, 20% purple skittles for hamburgers, 10% green skittles for salad, 20% yellow skittles for home lunch and 10% orange skittles representing no lunch.

Mrs. Moulton then has students randomly select 10 skittles at a time, repeated roughly five times, returning each group of 10 skittles to the bag each time. Notice that expected values are given. Because approximately 40% of the skittles in the bag are red, we should expect to average a draw of 4 skittles. The goal of the activity is that students notice that a random sampling gives close to 40% red skittles.

It is not necessary to replace each skittle individually after it is sampled. In sampling, if less than 10% of the population is sampled at a time, you do not have to sample with replacement. In this case, sampling 10 skittles will not change the ratio of, i.e., pizza:salad enough to matter. If you are sampling less than 10% of the population, and the sample is random, then independence can be safely assumed. This is called the 10% rule for sampling without replacement. Ten skittles are far less than 10% of the skittles in the bag; given the number of skittles in each bag are 500. Thus, samples of up to 50 without replacement would still be appropriate.

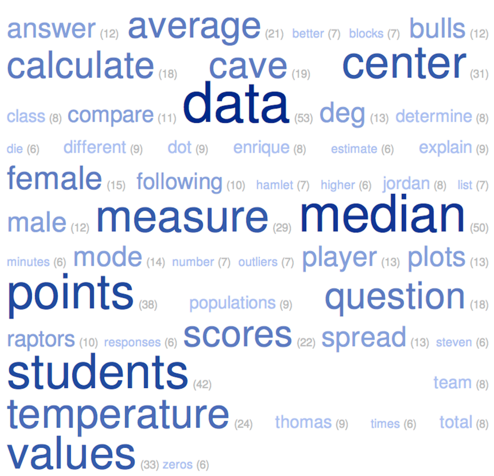
Let's consider one more scenario. Suppose Mrs. Moulton wishes to investigate the occurrence of pizza demand in her entire school district. A cluster sample could be taken by identifying the different school boundaries in her school district as clusters. Cluster sampling is a sampling technique where the entire population is divided into groups, or clusters, and a random sample of these clusters are selected. All observations in the selected clusters are included in the sample.

A sample of these school boundaries (clusters) would then be chosen at random, so all schools in those school boundaries selected would be included in the sample. It can be seen here then that it is easier to call, email, or visit several schools in the same chosen school boundary, than it is to travel to each school in a random sample to observe the occurrence of say pizza demand in the entire school district. The big question is: how is the demand for pizza at Mrs. Moulton's school similar or different from that of the school district as a whole?

Scientists use data from samples in order to make conclusions about the world. If using data from the samples, to come to an agreement on an estimate for the demand for pizza in the school district is calculated by averages, we give a cautionary note. If the sample sizes are different, then averaging the data gives more "weight" to the larger samples, so a method will need to be developed to come up with a way to adjust for the different sample sizes, that is, scaling it up to represent the population of the school district, such as multiplying the estimate by number of school boundaries in the district.

There potentially could be variability between all the samples taken, i.e. samples taken from a high school versus an elementary school. Variability is a measure of how much samples or data differ from each other. How could we accommodate for the variability? We would sample only schools that are most similar, i.e. just junior high schools within the chosen school boundary.

Sometimes data are examined to make a table of frequency of the entries. For example, if we want to study the height of 7th graders, we might collect data (from sufficiently large samples, and make a table showing the percentage of 7th graders in a given height range (say, counting by inches). In order to get a good estimate, we might take several samples of the same size. Another use is demonstrated in the course workbook (see section 7.2c of Chapter 7) of frequency of letters of the alphabet . Literary sites sometimes use word frequency of text to try to identify the author of the text, based on their knowledge of the word use of a collection of authors. A very nice piece of software allowing for quick visualization of use words is www.tagcrowd.com. A graphic of the 50 most common words in the first half of the workbook section 7.3, is displayed below.



Section 7.3: Draw Informal Comparative Inferences about two Populations

Informally assess the degree of visual overlap of two numerical data distributions with similar variabilities, measuring the difference between the centers by expressing it as a multiple of a measure of variability. For example, the mean height of players on the basketball team is 10 cm greater than the mean height of players on the soccer team, about twice the variability (mean absolute deviation) on either team; on a dot plot, the separation between the two distributions of heights is noticeable. 7.SP.3

Use measures of center and measures of variability for numerical data from random samples to draw informal comparative inferences about two populations. For example, decide whether the words in a chapter of a seventh-grade science book are generally longer than the words in a chapter of a fourth-grade science book. 7.SP.4

The focus of 7.SP.3 and 7.SP.4 is informal comparative inferences about two populations. In 7.SP.3 we informally assess the degree of visual overlap of two numerical data distributions with similar variabilities, measuring the difference between the centers by expressing it as a multiple of measure of variability. Practical problems dealing with measures of center are comparative in nature, as in comparing average scores on the first and second exams. Such comparisons lead to conjectures about population parameters and constructing arguments based on data to support the conjectures. If measurements of the population are known, no sampling is necessary and data comparisons involve the calculated measures of center. Even then, students should consider variability.

Specifically, students will calculate measures of center and spread from data sets, and then use those measures to make comparisons between populations and conclusions about differences between the populations.

'Data are readily available online with sources such as the American Fact Finder (Census Bureau), Statistical Universe (LEXIS-NEXIS), Federal Statistics (FedStats), National Center for Health Workforce Analysis (HRSA), CDC Data and Statistics Page, CIA World Factbook, and locally at Utah Division of Wildlife Resources (UDWR), USGS Water Data for Utah, Utah Statistics (aecf.org) or NSA Utah Data Center.

Research into data sets provides opportunities to connect mathematics to student interests and other academic subjects, utilizing statistic functions in promethean boards, graphing calculators, or excel spreadsheets; most especially for calculations with large data sets. In 6th grade the measures of central tendency (mean, median, mode) were studied, as well as various techniques for summarizing the variability in the data (dot plots, five-number summary and box plots). Here we will introduce a numerical measure of spread: the mean average deviation (MAD). This concept is a little more intuitive and easier to calculate (by hand) than the standard deviation that will be discussed in grade 10. To illustrate the use of these concepts we will work with a particular data set over the next few pages.

Example 18.

How much taller are the Utah Jazz basketball players than the students in Mr. Spencer's A3, 7th grade math class?

We will use this context over the next few pages as a vehicle to recall several important statistical measures of spread of data, and further developing their use started in 6th grade. These are: dot plots, histograms, five-number summary, boxplots and the mean average deviation (MAD).

Mr. Spencer wanted to compare the mean height of the players on his favorite basketball team (the Utah Jazz) and his A3 7th grade students in his mathematics class. He knows that the mean height of the players on the basketball team will be greater but doesn't know how much greater. He also wonders if the variability of heights of the Jazz players and the heights of the 7th graders in his A3 class is related to their respective ages. He thinks there will be a greater variability in the heights of the 7th graders (due to the fact that they are ages 12-13 and experiencing growth spurts) as compared to the basketball players. To obtain the data sets, Mr. Spencer used the online roster and player statistics from the team website, and the heights of his students in his A3 class to generate the following lists and then asked the following questions. First, the data:

Utah Jazz 2013-14 Team Roster: Height of Players in inches
(Data pulled from www.nba.com/jazzroster):

84, 73, 77, 75, 81, 81, 82, 84, 78, 80, 78, 75, 79, 83, 83, 71, 73, 81, 78, 81

Mr. Spencer's 7th grade A3 Class: Height of Students Fall 2013
(Data pulled from www.CDC.gov/growthcharts):

66, 65, 57, 56, 55, 54, 64, 71, 64, 58, 59, 56, 65, 65, 66, 65, 64, 65, 56, 73, 56, 64, 63, 60, 60, 55, 54, 60

a. To compare the data sets, create two dot plots on the same scale with the shortest individual 54 inches and the tallest individual 84 inches. A dot plot (Figure 1), is a graph for numerical variables where the individual values are plotted as dots (or other symbols, such as x). In contrast a histogram (Figure 2) is a graphical display of data using bars of different heights. There are many tools available to create both dot plots and histograms, such as promethean boards, graphing calculators, Excel spreadsheets, or in the case of the histograms, Geogebra. In many of the tools available in creating a histogram, a choice is made to determine the number of bins. In Figure 2, the choice was made to create 8 bins, but the number of bins can vary depending on the best way to summarize the data.

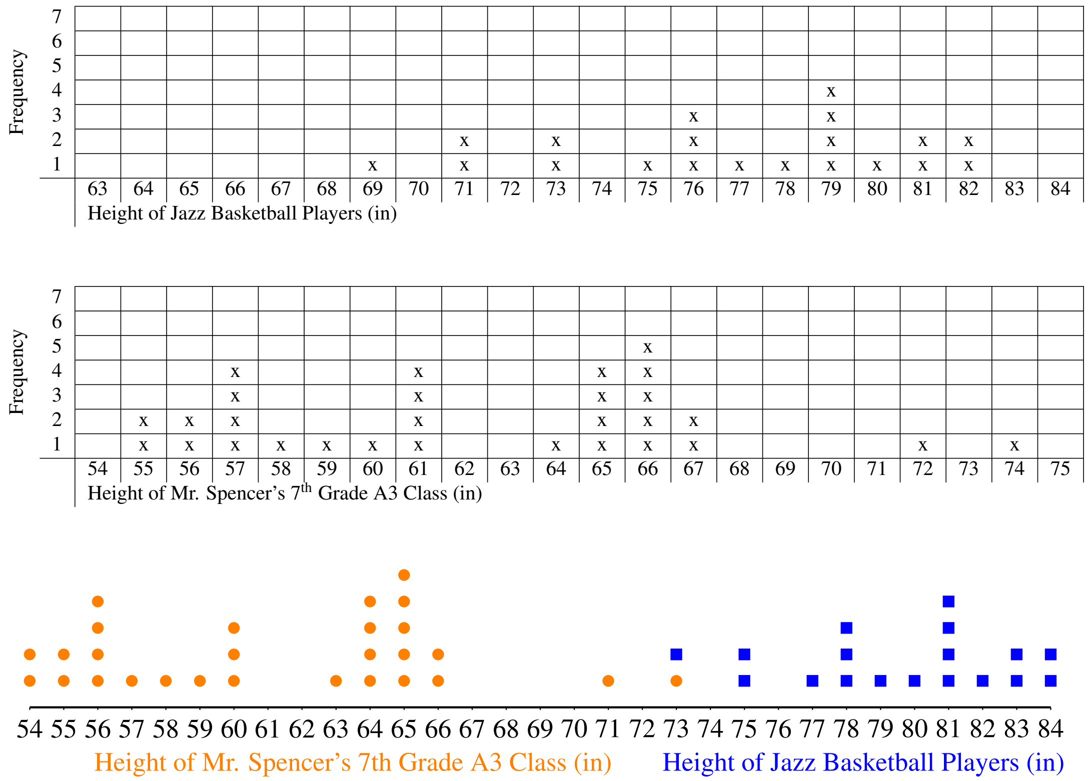


Figure 1

When we have a collection of numerical data it is especially helpful to know ways to determine the nature of the data. In particular, it is helpful to have a single number that summarizes the data. We are often interested in the measure of center and we commonly use the terms mean, median and mode as descriptors of the data. The mean, median and mode each provide a single-number summary of a set of numerical data; although we typically use the mean to make fair comparisons between two data sets. The mean is used for relatively normal data. The median is used for skewed distributions. It is not

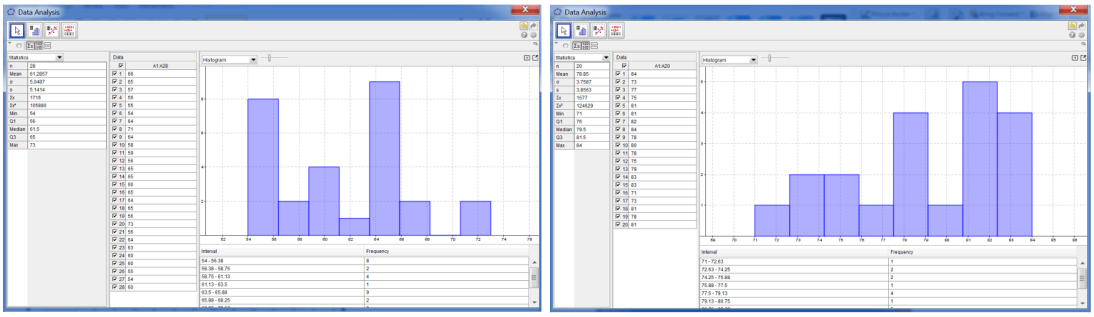


Figure 2

uncommon to have extreme values that will alter the mean of your data. In these situations, the median is a better measure of center. (see Figure 3). Mode is rarely used as a description for the measure of center.

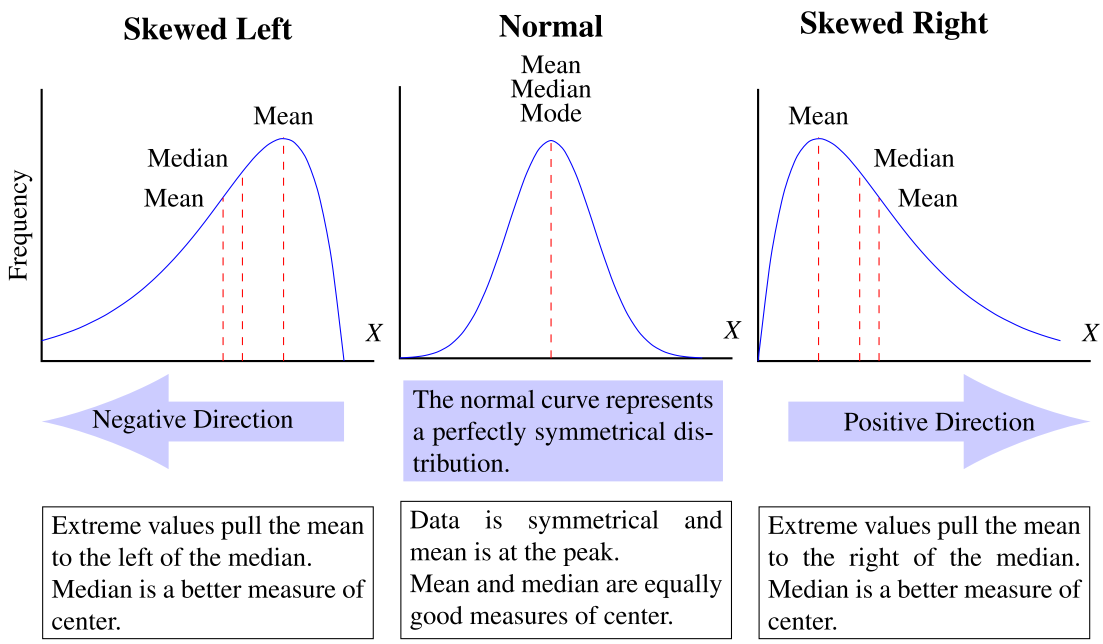


Figure 3

To calculate the mean, or the average, of a list of numbers, add all the numbers and divide this sum by the number of values in the list. For example, consider the data set {4, 9, 3, 6, 5}. The mean is

$$\frac{4+9+3+6+5}{5} = \frac{27}{5} = 5.4\,.$$

The mean is an important statistic for a set of numerical data: it gives some sense of the "center " of the data set. Two other such statistics, the median and the mode were discussed in 6th grade, but will not be considered here. A brief review is found in Section 7.3a homework.

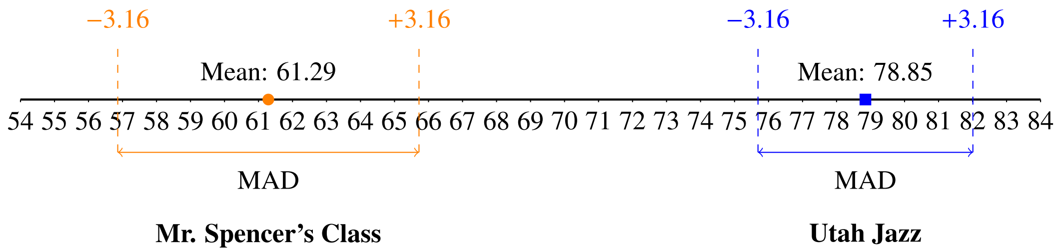
Once we have calculated the mean for a set of data, we want to have some sense of how the data are arranged around the mean: are they bunched up close to the mean, or are they spread out? There are several measures of the spread of data; here we concentrate on the mean absolute deviation (MAD). Methods for calculating center and MAD are standards from the Grade 6 curriculum and are being revisited in Grade 7 as they compare centers and spreads of different data sets. The mean absolute deviation (MAD) is calculated this way: for each data point, calculate its distance from the mean. Now the MAD is the mean of this new set of numbers. Let's do this calculation for the above set of numbers {4, 9, 3, 6, 5}, with mean 5.4.

Data Point	Mean	Deviation
4	5.4	1.4
9	5.4	3.6
3	5.4	2.4
6	5.4	0.6
5	5.4	0.4

Add the deviations: 1.4+3.6+2.4+0.6+0.4 = 8.4, and divide by the number of data points, 5, to get the MAD = 8.4/5 = 1.68.

Now, let's apply this technique to the two sets of data Mr. Spencer wants his class to consider. Before starting the computation Mr. Spencer asks, based on the representations in Figure 1 above, which of the data sets seems to have a larger mean absolute deviation (that is, a broader spread).

b. Once the MAD has been calculated, Mr. Spencer asks the class to put the means of the two data sets, and make marks of the place on both sides of the mean that are the MAD away from the mean. These insertions tell us directly which data set has the greater spread. The following image is the result of this work.



This display of the data confirms Mr. Spencer's hunch: that the data for the Utah Jazz are not spread out as much as the data for his class, and that the spread in his class is symmetric around the mean, while it is more spread in the lower end for the Jazz. What we see in the dot plot that we do not see in the MAD data is the two clusters of heights in Mr. Spencer's class.

When we use the mean and the MAD to summarize a data set, the mean tells us what is typical or representative for the data and the MAD tells us how spread out the data are. The MAD tells us how much each score, on average, deviates from the mean, so the greater the MAD, the more spread out the data are.

A *box plot* (or box-and-whisker plot) is a visual representation of the *five-number summary*, and tells us much more about the spread of the data, answering questions like: on which side of the mean is there more spread; how far away the extremes are. Recall these statistics from Grade 6. First, the *median* is the middle number: there are as many values below the median as there are above. The five number summary shows 1) the location of the lowest data value, 2) the 25th percentile (first quartile) or the center between the minimum and the median, 3) the 50th percentile (median), 4) the 75th percentile (third quartile) or the center between the median and the maximum, and 5) the highest data value. A box is drawn from the 25th percentile to the 75th percentile, and 'whiskers' are drawn from the lowest data value to the 25th percentile and from the 75th percentile to the highest data value. If there is no single number in the middle of the list, the median is halfway between the two middle numbers.

As an example we find the five-number summary for the data set: Height of the Utah Jazz Basketball Players 2013-14 season.

Median is the average of 79 and 80; (79 + 80) 2 = 79.5

71 73 73 75 $\underbrace{75 \quad 77}_{\text{Quartile 1}}$ 78 78 78 $\underbrace{79 \quad 80}_{\text{Median}}$ 81 81 81 $\underbrace{81 \quad 82}_{\text{Quartile 3}}$ 83 83 84 84

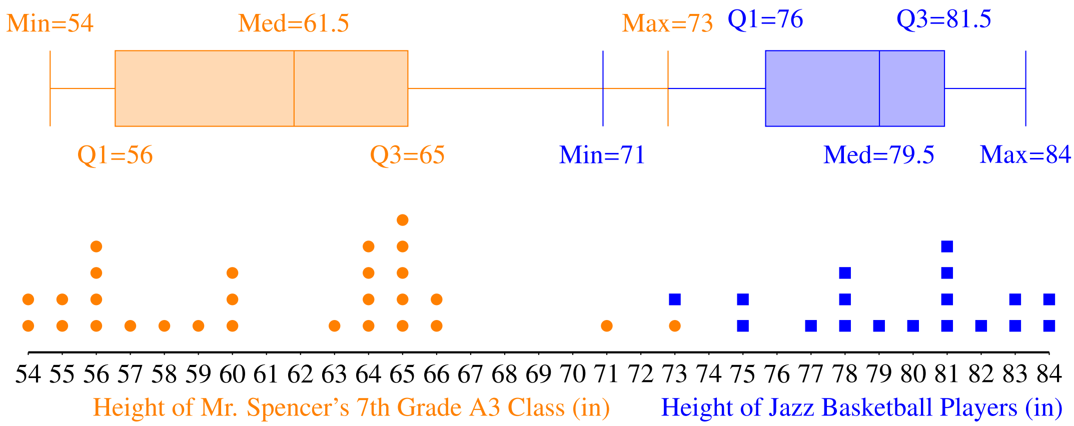
Quartile 1 (Q1) is the average of 75 and 77; (75 + 77) 2 = 76

Quartile 3 (Q3) is the average of 81 and 82; (81 + 82) 2 = 81.5

The five-number summary is:

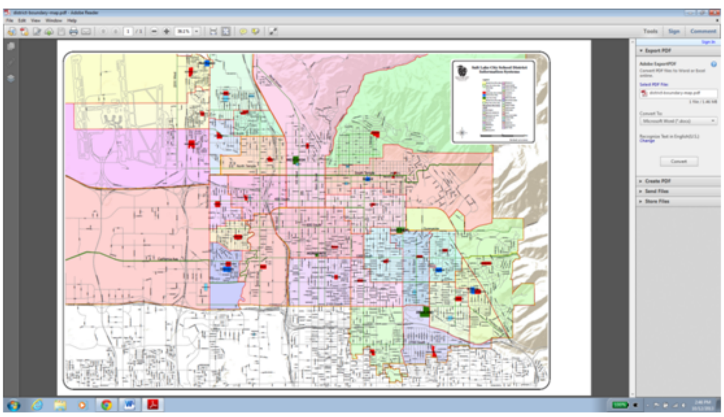
Min	Q1	Median	Q3	Max
71	76	79.5	81.5	84

Given the five number summary, we can make the box-plots. Here we show the box plots for both sets of data Mr. Spencer wants to compare, and below that the dot plots so that we can compare the information that can be obtained from each representation. For example, the box plots show greater spread in Mr. Spencer's class, and that in both cases the spread of the second quartile is greater than the spread of the third quartile.

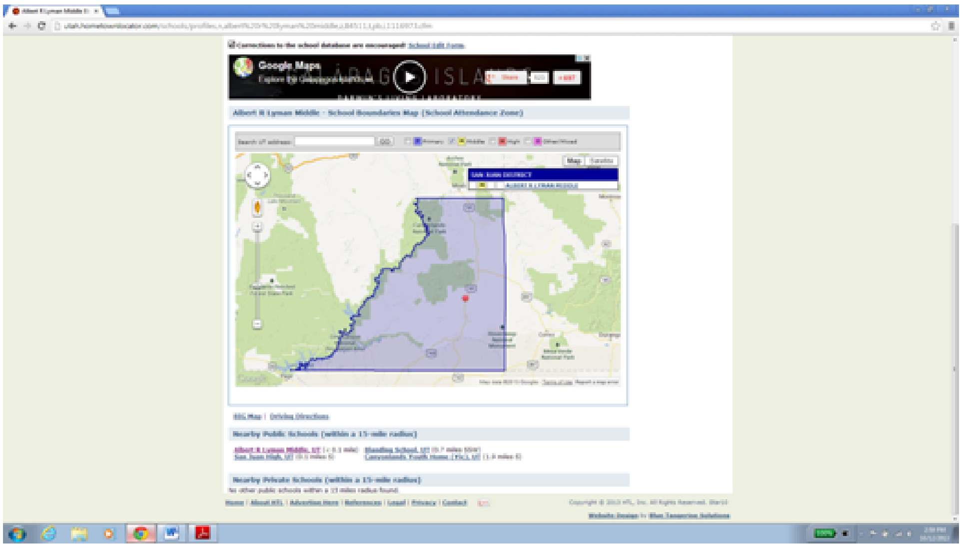


Example 19.

Glendale Middle School is located in the heart of Salt lake School District. It is one of five middle schools in the district, and has approximately 835 students recorded in attendance, (data from the 2009-2010 school year). The map shows the boundary of the Salt lake School District, and the arrow pointing to the small blue area is the boundary of the middle school, and the region for the Glendale Middle School is the blue area denoted by the arrow.



Albert R. Lyman Middle School is located in the San Juan School district. In the 2009-2010 school year there were approximately 312 students in attendance. The school is located in Blanding, Utah and is the only middle school in the district. The blue portion highlighted is the San Juan School District.



The State School Board wants to determine how far students travel to school and picked two schools; Albert R. Lyman Middle in San Juan School District and Glendale Middle in Salt Lake City School District. Ten students each, at both schools, were chosen at random and were asked how far they traveled to school. The responses are below:

Glendale Middle
0.1
0.3
0.4
0.6
0.7
0.8
1.2
1.6
2.8
5

Albert R. Lyman Middle
0.5
1.5
4
5
10
12
18
24
30
65

The State School Board asked the students to answer the following questions.

a. What is the mean of "distance traveled" for each school, and what does the mean represent?

b. What is the mean absolute deviation (MAD) for both schools? Create a table for each data set to help with the calculations. Describe what the mean absolute deviation represents?

c. To compare the data sets, create two dots plots on the same scale. What conclusions can be made from these two data sets? Note: We are only working with data from 10 students and conclusions need to be cautiously represented. Conclusions about students at the respective schools cannot be made without having an adequate sample size and confirming that the students were chosen at random.

Summary

This unit covers the importance of randomness in sampling, and of using samples to draw inferences about populations. The statistical tools introduced and practiced in 6th grade were reinforced and expanded upon as students continued to work with measures of center and spread to make comparisons between populations. Students will have investigated chance processes as they develop, use, and evaluate probability models. Compound events were explored through simulation, and by multiple representations such as tables, lists, and tree diagrams.

The eighth grade statistical curriculum will focus on scatter plots and bivariate measurement data. Bivariate data is also explored in Secondary Math 1, however, Secondary Math I, II, and III statistics standards return to the exploration of center, variance and distribution, random probability calculations, sampling and inference.

Chapter 8
Measurement in 2-3 Dimensions, Cross-Sections of Solids

Geometric and spatial thinking connect mathematics with the physical world and play an important role in modeling phenomena whose origins are not necessarily "physical.'" An example of this is the use in 6th grade of *Nets* in the context of area and volume. Geometric thinking is also important because it supports the development of number and arithmetic concepts and skills, by providing students with a context for intuitive understanding. The sections in this chapter emphasize key ideas that assist students in developing a deeper understanding of numbers. In grades K-6 students learned to work with basic two-dimensional geometric shapes: triangles, squares, rectangles, and others. In addition, students learned specific parts and properties of shapes allowing them to identify and classify them into categories, and to determine how the categories of shapes are related. In this chapter students will be engaged in using what they have previously learned about drawing geometric figures (using rulers and protractor, coordinate grids and technology) to solve problems involving area and circumference of a circle, and real-world mathematical problems involving area and perimeter of two-dimensional objects composed of triangles and quadrilaterals. Furthermore, students will explore 3D geometric figures and circles and apply their mathematical knowledge of rational numbers, proportionality and algebra to solve problems involving surface areas and volumes, and to express meaningful formulas and recognize equivalent expressions.

More specifically, section 8.1 builds from understandings of geometry, measurement and data from grades 3-6. It utilizes the scope of the number system and is a review and extension of previously learned skills. For example, in sixth grade students learned how to find area by creating rectangular arrays. Using the shape composition and decomposition skills, students learned to develop area formulas for parallelograms and triangles. They also learned how to address three different cases for triangles: a height that is a side of a right angle, a height that lies over the base and a height that is outside the triangle. Composition and decomposition of regions continues to be important for solving a wide variety of area problems, including justifications of formulas and solving real world problems, as we will see in section 8.1. We will further see that composition and decomposition of shapes is important since it is used throughout geometry from Grade 6 to high school and beyond.

Previously, in Chapter 5, students learned how to find the circumference and area of circles, whereas the focus of section 8.1 will be to extend and apply the area and perimeter of circles, triangles, rectangles, parallelograms, and trapezoids to various real-world and mathematical problems. Our goals for section 8.1 will be: i) solving problems involving area and circumference of a circle, ii) solving real-world and mathematical problems involving area and perimeter of two-dimensional objects composed of triangles and quadrilaterals, yet most importantly contrasting and relating area and perimeter.

Our focus for section 8.2 will center on 3D figures. Students begin by examining plane sections of 3D figures. The point of work in the elementary grades with plane sections was to develop the ability to use drawings and physical models to identify parallel lines and planes in 3D shapes, as well as lines perpendicular to a plane, lines parallel to a plane, and to be able to construct the plane passing through three given points, and the plane perpendicular to a given line at a given point. For this reason, in the elementary grades, plane sections were actually *cross sections*: special plane sections parallel to a face of the object, or perpendicular to an axis of symmetry of the object. (We note that it has become customary to use these names interchangeably). In grade 8 we want to go more deeply in the detailed visualization of 3D objects, and for that reason, we consider all sorts of plane sections.

Furthermore, in the elementary grades, students study volume and surface area of special objects in a descriptive way. In 7th grade we want to go further, in order to understand the distinctions and relations between surface area and volume. As the volume of an object grows, does its surface area grow? This is the analog in 3D of the study of perimeter and area of figures in the plane. Here we introduce the ideas involved in computation of volumes, and then relate that to the determination of surface area using nets (as in 6th grade). Students will then differentiate between surface area and volume and use their understanding to solve various problems.

One of the tools introduced at this point is Cavalieri's principle: that the volume of a figure developed around a particular axis is determined by the area of the section of the object by planes perpendicular to the axis. This is not a grade 8 core topic, but it seems to fit naturally and easily in the discussion of sections, to provide an added intuition into area calculations.

Section 8.1. Measurement in Two Dimensions

Solve real-world and mathematical problems involving area, volume and surface area of two- and three- dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms. 7.G.6.

Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle. 7.G.4.

Throughout this chapter, as in Chapter 5, students and teachers use geometric terms and definitions with which they have becomes familiar: polygons, perimeter, area, volume and surface area of two-dimensional and three-dimensional objects, etc. Though these terms are not rigorously defined, it is important that they are used correctly and misconceptions are not allowed to develop. For this reason we start by reviewing the frame for using geometric terms. Something that we cannot stress too much is that to "know the formula" does not mean memorization of the formula. We are striving for an understanding of why the formula works and how the formula relates to measure (length, area and volume) and the figure. The surface area formulas are not the expectation with this standard; the expectation is that students will understand the process and procedures for determining those formulas.

A central construction of objects in three dimensions is that of drawing a planar figure out in the third dimension. This creates the parallel with two dimensions: just as area in 2D is the product of length and the distance this length is drawn out, volume in 3D is the product of area with the distance drawn.

A *polygon* is a planar figure consisting of a sequence of line segments with the property that the initial point of each line segment is the end point of the previous line segment, and the endpoint of the last segment is the initial point of the first segment. These endpoints are the *vertices* of the polygon. A *triangle* is a polygon with three sides, and a *quadrilateral* has four sides. Before moving on to more detailed vocabulary, it is necessary to point out an ambiguity, which in this text will be resolved by the context. By this definition, a triangle consists of the set of line segments that act as the boundary of a region in the plane. When we speak of the *area* of a triangle, we mean the area of the region bounded by the triangle. Similarly *circle* - the set of all points equidistant from one point, called its *center* - refers to the boundary; and when we speak of the area of the circle, we mean the area of the region bounded by the circle (to which we sometimes refer as the *disk* bounded by the circle).

Now, different kinds of triangles are defined by adjectives: acute, scalene, right, isosceles, etc. But for quadrilaterals, we have different nouns: square, rectangle, parallelogram, etc. This issue arises: when we say the word "rectangle" do we mean a rectangle that is not a square, or are squares included? Ordinarily, the designations are meant to be included: a square is a particular kind of rectangle. For clarity, let's define the various kinds of quadrilaterals, starting from the most inclusive:

- A *quadrilateral* is a polygon with four sides.

- A *trapezoid* is a quadrilateral with one pair of parallel sides .
- A *parallelogram* is a quadrilateral with both pairs of opposing sides parallel.
- A *rhombus* is a parallelogram with all sides of the same length.
- A *kite* is a quadrilateral with two pairs of adjacent sides of the same lengths.
- A *rectangle* is a parallelogram with at least one right angle.
- A *square* is a rectangle with all sides of the same length.

The following diagram shows the relationships among these figures. See Figure 1 for images of these different categories of polygons.

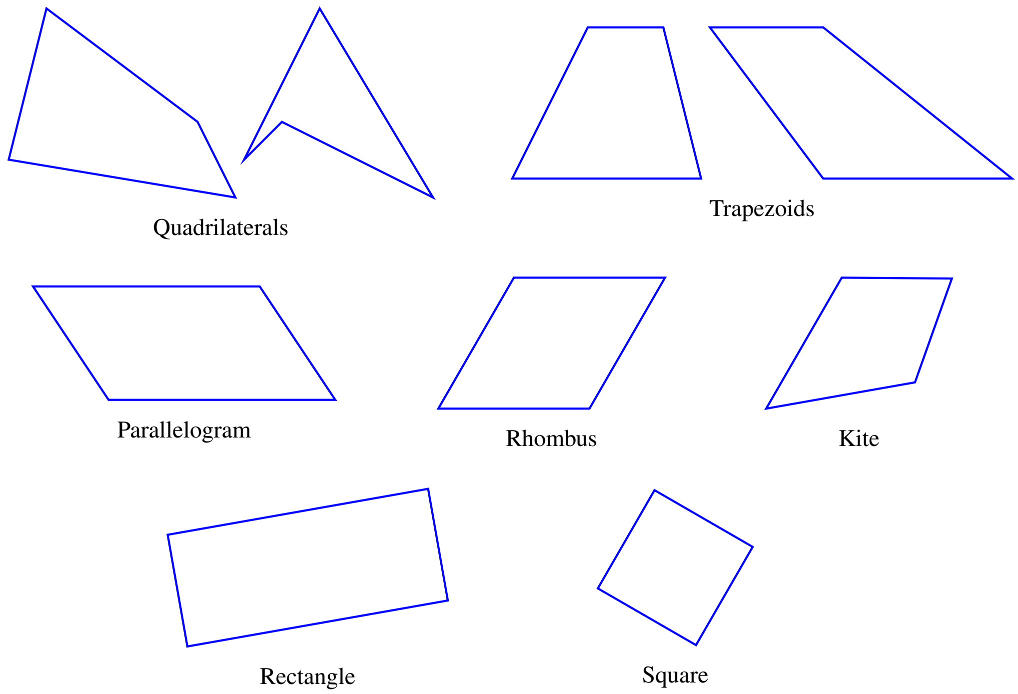
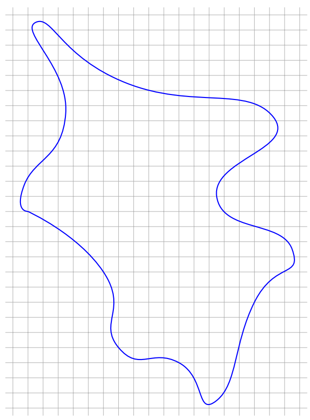


Figure 1

Area



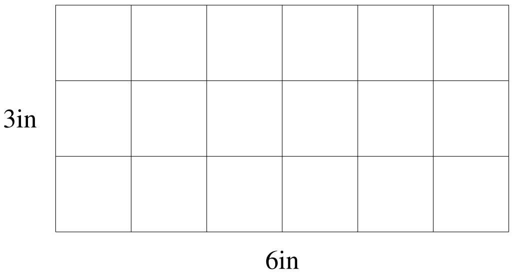
In two-dimensional space, area measures the space enclosed by a figure. The calculation of area of land is one of the oldest problems in mathematics, originating in part from problems of fair apportionment of land inherited by children from their parents. The land itself may include parts of rivers or edges of foothills or swamps, small bodies of water, or hilly regions. One method of finding the area of irregular pieces of land is not very far from the approach taken in calculus; a surveying team will make the relevant linear measurements of the boundary, and then make a scale drawing. Or, the land is photographed from above, thus making it flat. A grid of squares of the same dimensions is placed over the drawing or the photo.

The number of squares completely inside the boundary plus half the number crossed by the boundary provides an estimate for the area. If a more precise estimate is desired, then smaller squares are used. This is the approach we will take in developing and discussing the area of plane figures, which also applies to the calculation of the surface area of common three-dimensional figures, using nets as in sixth grade.

That is, we will trace everything ultimately back to the area of squares, because we measure area in square units. In real life problems, the unit square could be one inch, one centimeter, one yard, or even one mile on each side, depending on what you are measuring.

Now, length is a measure of pieces of lines, and so area is to be thought of as a measure of pieces of a surface, one that gives the planar or surface "content" of a figure.

How do we figure out the area of a polygonal shape? What does it mean to say that the area of a region is 18 square inches? It means that the shape can be covered, without gaps or overlaps, with a total of eighteen 1-inch-by-1-inch squares, allowing for squares to be cut apart and pieces to be moved if necessary. If the figure is a 3×6 rectangle, we can cover it with 1×1 squares, and count the squares: there are 18 of them. In fact, students will recall that this is the geometric intuition that led to the concept of multiplication: 3×6 *is* the area of a rectangle with side lengths of 3 units and 6 units.



3 rows with 6 squares, each of area $1in$, in each row, so the area is $3in \times 6in$

How about a general polygonal figure? In general, the technique for calculating areas of general polygonal figures, or formulas for specific types of polygons, is based on these principles:

1. If you move a shape rigidly (without stretching or distorting it), then its area does not change.
2. If you combine (a finite number of) shapes without overlapping them, then the area of the resulting shape is the sum of the areas of the individual shapes.

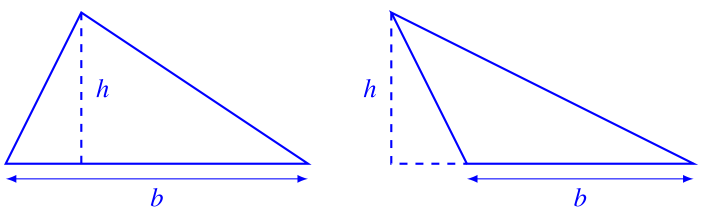
In general a polygonal figure can be partitioned into combination of simpler shapes (actually, triangles and rectangles) for whose areas formulas have been developed already. Then the area of the original figure is the sum of the areas of the component figures. We will summarize this development in the following sequence of examples.

Example 1. Rectangle.

If the lengths of the sides of a rectangle are a and b units, then its area is ab square units.

Example 2. Triangle.

To find the area of a triangle: designate one its sides as the *base*, and denote its length by b. The distance from the base to the opposing vertex is called the *height* of the triangle (see Figure 2). Then the area of the triangle is $\frac{1}{2}bh$.



Acute Triangle   Obtuse Triangle

If the triangle is a right triangle, we can designate one leg to be the base (of length b) and the other to be the height (of length a). If we rotate the triangle around its hypotenuse, we obtain a rectangle consisting of two copies of the given triangle, whose area is bh (see Figure 3), so the area of the triangle is half that.

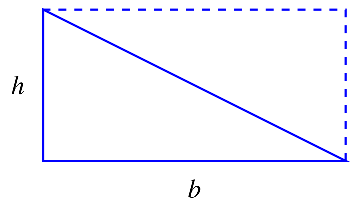


Figure 3

Now, referring back to Figure 2, when we drop the perpendicular we divide the triangle into two right triangles, the sum (or difference, depending upon whether the original triangle is acute or obtuse) of whose bases is b, so the general fact holds.

Example 3. Parallelogram.

Choose one side of the parallelogram, call it the *base* and its length b, and let h be the distance between the base and its opposite side. (See Figure 4). Then, the area of the parallelogram is bh.

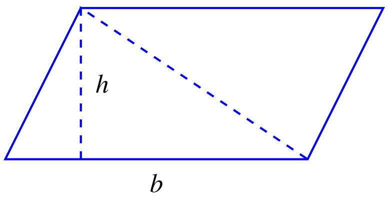


Figure 4

If we now draw a diagonal of the parallelogram, it divides the parallelogram into two triangles, each of which has area $\frac{1}{2}bh$, so the result follows.

Example 4. Trapezoid.

Let the lengths of the parallel sides be b_1 and b_2, and the distance between them h. Then the area of the trapezoid is $\frac{1}{2}(b_1 + b_2)h$.

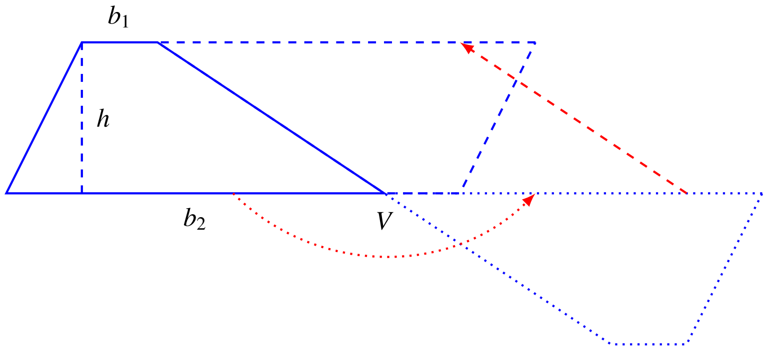


Figure 5

Rotate the trapezoid around the vertex V for 180°, as shown by the red circular arrow. Now translate the new trapezoid as indicated by the straight red arrow, so that the lower bases of the two trapezoids are on the same line. The result is a parallelogram of height h and side length $b_1 + b_2$. Thus its area is $(b_1 + b_2)h$). Finally, the original trapezoid is precisely half of this parallelogram, and so its area is $\frac{1}{2}(b_1 + b_2)h$.

Example 5.

Find the areas of the polygon in Figure 6, given that the length of the side of the square in the grid is 1 cm.

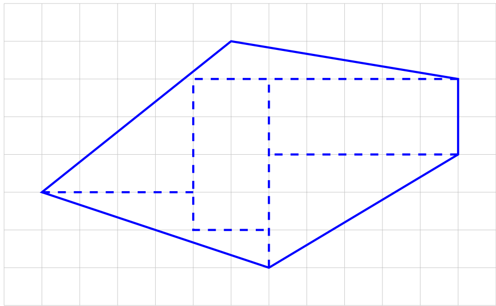


Figure 6

Solution. This figure consists of two rectangles and 4 triangles.

- central rectangle: area is 3 sq. cm.
- right rectangle: area is 10 sq. cm.
- upper left triangle: area is 10 sq. cm.
- upper right triangle: area is 3 sq. cm.
- lower left triangle: area is 6 sq. cm.
- lower left triangle: area is 7.5 sq. cm.

The sum is the area of the entire figure: 39.5 sq. cm.

To complete our list of fundamental figures, we include the circle, discussed in Chapter 5.

Example 6.

The area of a circle of radius r is $A = \pi r^2$, where π is approximately 22/7.

Example 7.

A 14 in. pizza has the same thickness as a 10 in. pizza. How many times more ingredients are there on the larger pizza?

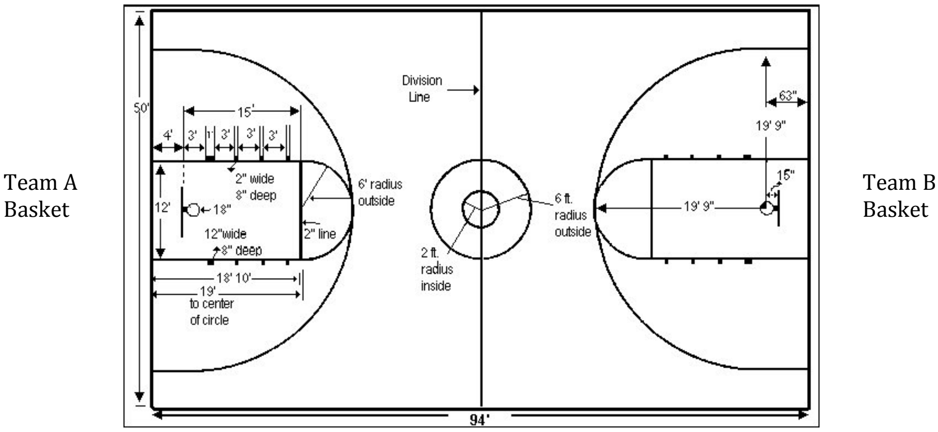
Solution. Pizzas are measured by their diameters, so the radii of the two pizzas are 7 in. and 5 in., respectively. Since the thicknesses are the same, the amount of ingredients used is proportional to the areas of the pizzas. The larger pizza has area $\pi 7^2 = 49\pi$ sq. in., and the smaller pizza has area $\pi 5^2 = 25\pi$ sq. in. The ratio of areas is

$$\frac{49\pi}{25\pi} = 1.96\ ,$$

so a 14 in. pizza has about twice the ingredients of the 10 in. one.

Example 8.

The three-point line in basketball is approximately a semi-circle with a radius of 19 feet and 9 inches. The entire court is 50 feet by 94 feet. What is the area of the court that results in 3 points for Team A (given Team A is shooting towards its basket)?



SOLUTION. The three-point line is the line that separates the two-point area from the three-point area; any shot converted beyond this line counts as three points. First we decompose the area into shapes that we know well, a semi-circle and a rectangle. Then we find the area of each shape. To determine the total three-point area for Team A, we subtract the total area of the semicircle from the total area of the rectangle:

Area of rectangle = 50 ft $\times$ 94 ft = 4700 sq ft;

Area of semicircle = $\frac{1}{2}\pi r^2 = \frac{1}{2}\pi 19.75^2 = 612.71$ sq ft.

Total area that will result in 3 points for Team A;

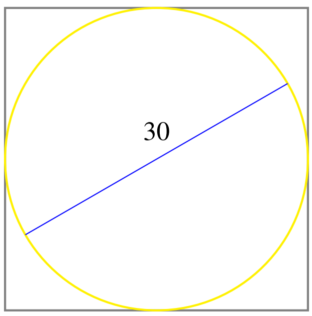
$$A_{rectangle} - A_{semicircle} = 4700 - 612.71 = 4087.29 \text{ sq. ft} \quad \text{approximately} .$$

Perimeter

The perimeter of any polygonal region is the sum of the lengths of its sides. So, the perimeter of a square of side length 5 in. is 20, since there are 4 sides. Contrast this to area, which is the product of the basis cimenstions; in this case 5-by-5, giving 25 sq. in.

EXAMPLE 9.

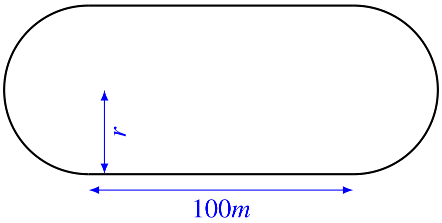
A ring made of gold that has a diameter of 30 cm is put in a silver display box so that the ring just fits. What is the length of the ring of gold? What is the length of the frame of silver?



SOLUTION. The radius of the ring of gold is 15 cm, so the length of the ring is $2\pi \times 15 = 94.2$ cm, approximately (here we have approximated the value of π by 3.14). The perimeter of the box of side length 30 cm is 120 cm. Thus we have about four-thirds as much silver as gold.

Example 10.

Typically, a (foot) race track is formed by a rectangle with a semicircle at the short ends, so that the total distance is 400 m (about one-quarter mile), and the straight lengths (the long sides of the rectangle) are 100 m each. What is the radius of the semicircle?



Solution. The perimeter of the track is 400 m; of these there are two 100 m straightaways and two semicircles of radius r (which is the same as a full circle of radius r). So, we must have $100+100+2\pi r = 400$. Solve for r to get $r = \frac{100}{\pi} = 31.86$ m.

Example 11.

Actually, a track consists of 6 or more lanes, each of which is 1.2 m wide. What we have just calculated is the inside length of the first lane. What is the inside length of the second lane?

Solution. The radius of the semicircle at the ends formed by the inside of the the second lane is 1.2 m longer than the inside length of the first lane, so is $31.86+1.2 = 33.06$ m. Then the total length around the track on the second lane is $100 + 100 + 2\pi(33.06) = 407.62$, and the second lane is thus 7.62 m longer than the first lane. In fact, each lane (on its inside) is 7.62 m longer than the preceding lane: this is why racers running a 400 m race start at intervals of separated by 7.62 m.

Contrasting and Relating Area and Perimeter

If you know the distance around a shape (its perimeter), can you determine its area? If you know the area of a shape, can you determine that distance around the shape? Does perimeter determine area? For a square, the answer is "yes": if you know the perimeter you can find the side length and from this, the area. For a circle, again the answer is "yes": if you know the perimeter (i.e., the circumference) you can find the radius and hence the area. Yet, they these are different types of measurements, and they are expressed using different units, and are calculated in two different ways. Perimeter is measured in units of length, and area is measured in square units of length. Furthermore, perimeter is calculated by adding lengths, and area is calculated by multiplying lengths appropriately. This is strictly true for rectangles: perimeter is the sum of all of the sides, and area is found by multiplying the two side lengths.

Example 12.

Calculate the area and the perimeter of a rectangle of side lengths 6 ft. and 4 ft. , and another rectangle of side lengths 8 ft. and 2 ft. See Figure 7.

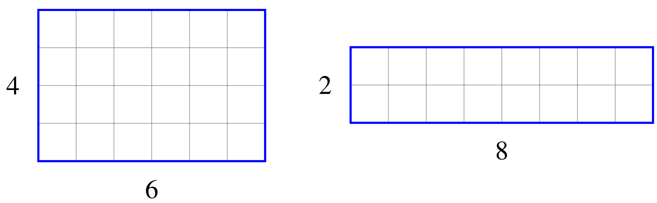


Figure 7

The areas are 24 sq. ft. and 16 sq. ft., but the perimeters of both rectangles are the same: 20 ft.

If we calculate the areas of many rectangles of the same perimeter, we see that the if the side lengths are closer in measure then the area of the rectangle is larger. That can be shown graphically for the two rectangles of Example 12. In Figure 8, notice that as the longer side of the rectangle is made shorter, the shorter side of the rectangle becomes longer to preserve the perimeter. The effect is to lose some area at the short end but gain more area on the long side of the rectangle for an overall increase in area. If we continue to decrease the length of the longer side and increase the length of the shorter side, we see the same phenomenon until we reach the rectangle with all sides the same length, 5. This gives strong evidence that, among all rectangles of given perimeter, the square has the largest area. Students should experiment with this statement with a variety of rectangles.

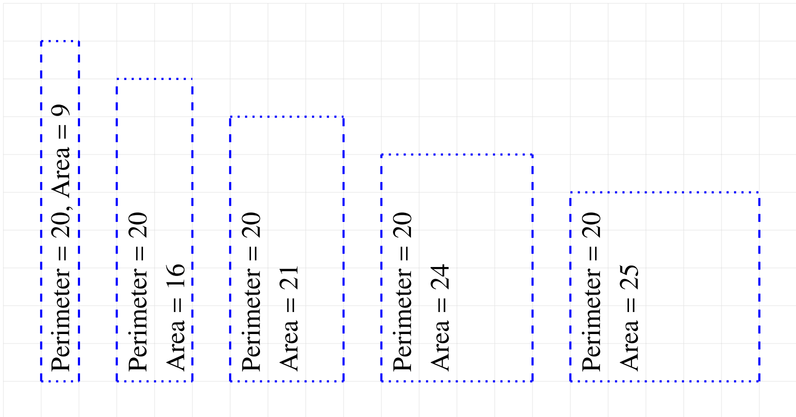


Figure 8

Example 13.

Let's follow through on this with another example: Among all rectangles of perimeter 50 units, find the rectangle with the largest area. Now, if the long side of the rectangle is 15 units, the shorter side has length 10 units, and the area is 150 square units. If the long side has length 20 units, the short side has length 5 units and the area is 100 square units. Here is a table of values of the area of a rectangle with a given long side:

Long side length (in units)	24	23	22	21	20	19	18	17	16	15	14	13	12
Area (in square units)	24	46	66	84	100	114	126	136	144	150	154	156	156

We have stopped the computation at the length 12 units, for from that point on this is not the length of the "long side." In fact, the square with the perimeter 50 units has side length 12.5 units and area 156.25 square units, and that is the rectangle with perimeter 50 units and greatest area. It is easier and more fun to experiment with this at websites set up for this purpose. For example,

http://www.mathopenref.com/triangleareaperim.html

is such a site in which area and perimeter are interactively explored with triangles.

Section 8.2. 2D Plane Sections of 3D Figures, 3D Measurement

Describe the two-dimensional figures that result from slicing three-dimensional figures, as in plane sections of right rectangular prisms and right rectangular pyramids. 7.G. 3.

Solve real-world and mathematical problems involving area, volume and surface area of two- and three- dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms. 7.G.6.

Now we turn to shapes in 3D and how to visualize them using our knowledge about 2D figures. Students begin by examining plane sections of 3D figures. A *plane section* of a solid in 3D is the 2 dimensional figure one gets by slicing the solid along some plane in space. In Figure 9a we show a plane section of a bagel.

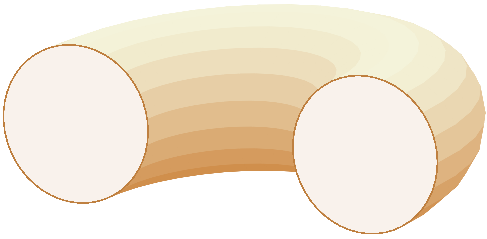


Figure 9a

This is not the usual section of a bagel, which will be along the plane of its major diameter. In that case, we get an image like this:

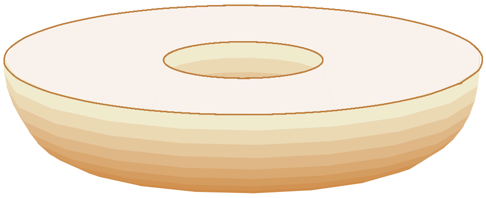


Figure 9b

In common language, a plane section is also called *cross section*. Often the name cross section is reserved for a section of a 3D object that is parallel to a particular plane of symmetry of the object, or perpendicular to a particular line of symmetry. For example, one line of symmetry for a cube is a line joining the centers of opposite faces, and a cross section perpendicular to that line is a square.

The emphases of this section are:

1. Describe the different ways to slice a 3D figure.
2. Describe the different 2D cross-sections that will result depending on how you slice the 3D figure.
3. Solve real-world and mathematical problems involving volume and surface area of three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.

Because one of the purposes of this section is to help students visualize and draw (or otherwise represent) three-dimensional figures, a brief review of some geometric terms and definitions is appropriate.A closed, connected shape in space whose outer surfaces consist of polygons such as triangles, squares, or pentagons is called a *polyhedron* (*polyhedra* is the plural). The polygons that make up the outer surface of the polyhedron are called the faces of the polyhedron. The line segment where two faces come together is called an edge of the polyhedron. A corner point where several faces come together is called a vertex (vertices is the plural) or corner of the polyhedron. The name polyhedron comes from the Greek; *poly* meaning "many" and *hedron* meaning "face," so polyhedron lierally means "many faces." Figure 11 depicts a typical polyhedron. This polyhedron has 10 vertices, 14 edges and 6 faces.

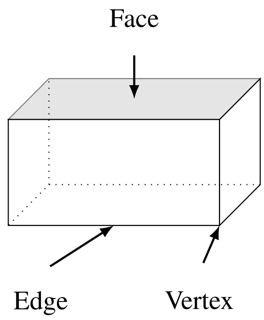


Figure 10

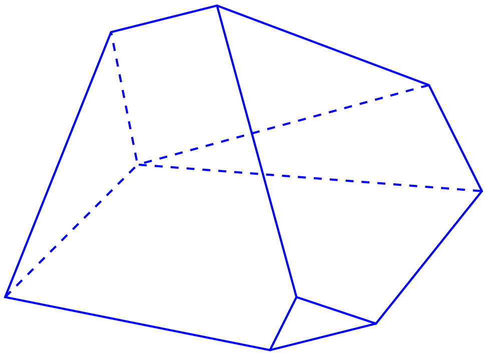


Figure 11

General Prisms

Start with two planes in space that are parallel. For any polygonal figure in one of the planes, sweep it through space in a direction perpendicular to the starting plane until we reach the ending plane. The resulting 3D figure is called a *prism*. In Figure 12 are three 3D figures obtained by sweeping a figure in the bottom plane out to the top plane. The first figure is a representation of a general prism. The second is called a *circular cylinder* (or, commonly, a *cylinder*). Since it is formed by sweeping out a circle and not a polygon, it is not a prism. The last is a prism, called the *triangular prism* or *wedge* since it is formed by sweeping out a triangle.

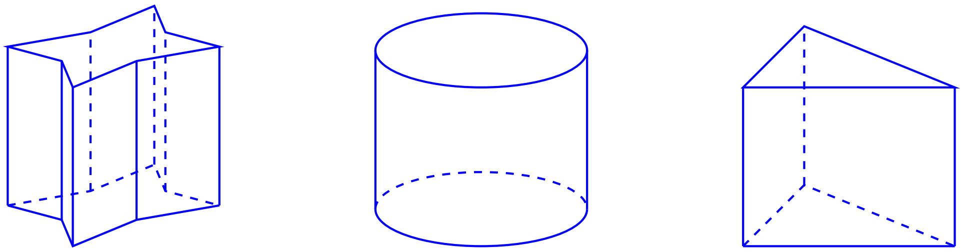
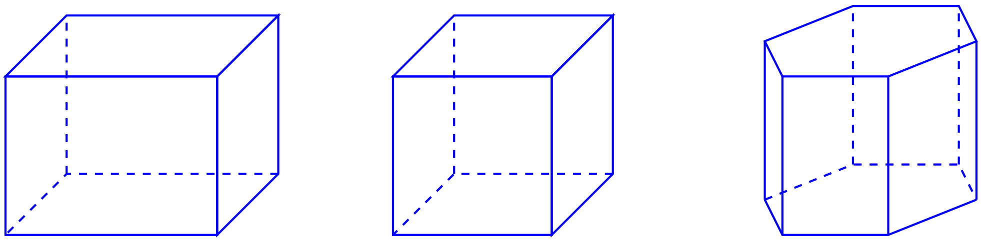


Figure 12

Technically, these should be called *right prisms*, with *right* specifying that these figures have been drawn out in a direction perpendicular to the plane of the start figure. Since, in 7th grade, we consider only this case, we will not use the adjective *right*. Notice also that any section of such a solid by a plane parallel to the original planes is a copy of the original planar figure. In Figure 13 there are three more solids of interest that are specific prisms:



Rectangular Prism

Cube

Hexagonal Prism

Figure 13

Note that a cube is a special kind of rectangular prism, one in which all edges are of the same length. For a prism, we shall refer to the planar figure from which it is drawn out as the *base*, and the distance it is drawn out as its *height*. In the first set of figures and the hexagonal prism, there is only one face that qualifies as the base; however, the rectangular prism can be realized as drawing out of any one of its faces, so the word "base" could be attributed to any one of its faces. In any problem, pick a face to be the base if it makes the problem simpler; otherwise it doesn't matter.

General Pyramids

Start a plane in three-dimensions and a point A not on the plane, and a polygonal figure F on the plane. Attach all points on A to$F$ by line segments. The resulting 3D figure is the *pyramid* w ith *base* F and *apex* A. For a polygonal figure in one of the planes, sweep it through space in a direction perpendicular to the starting plane until we reach the ending plane. The resulting 3D figure is called a *pyramid*. The first solid in Figure 14 is a *rectangular pyramid* because its base is a rectangle (probably a square) and the last solid is a *triangular pyramid* because its base is a triangle. This figure is also called a tetrahedron because it consists of four faces, all triangles. If the faces are all equilateral triangles, it s a *regular tetrahedron*. The middle figure, the *circular cone* also consists of all lines from a planar figure to a point not on the plane, but since the starting figure s not a polygon, it is not a pyramid.

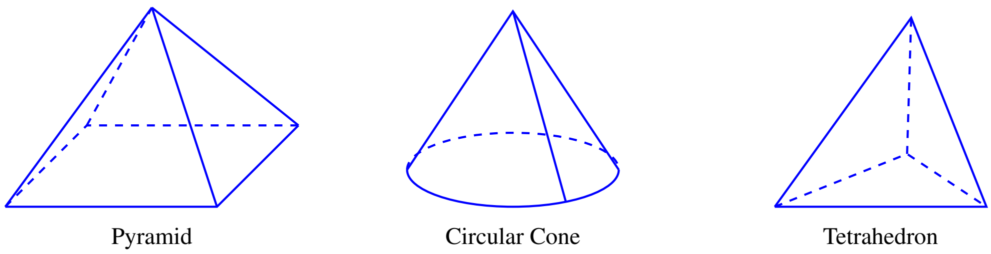


Figure 14

We shall be considering cones with the property that the line from the apex to the center of the base is perpendicular to the base (technically, these are called *right* cones). Note that sections of the cone by a plane parallel to the base produces a figure that is a scaled version of the base, with the scale factor reducing as we move toward the apex. In our figure for the pyramid, the base appears to be a square, but it need not be. The great pyramids in Egypt are all built above a square base, whereas many Mayan pyramids (in Mexico) have rectangular (non-square) bases.

2D Plane Sections of 3D Figures

We now focus on two-dimensional aspects of solid shapes: plane sections. Plane sections provide two-dimensional information about the inside of a shape. Thinking about cross-sections can help us recognize that solid shapes have an interior in addition to an outer surface. Even as we study volume in section 8.2, we can think of the volume of a prism as decomposed into layers, where each of these layers are much like thickened cross-sections. As mentioned earlier, when the sectioning plane is perpendicular to a particular axis of the figure, we will use the term *cross section*; it turns out that such sections are particularly important for prisms and cones.

Example 14.

What shapes can be created by one slice through a cube? Look for these possibilities:

a. a square

b. an equilateral triangle

c. a rectangle that is not a square

d. a triangle that is not equilateral

e. a pentagon

f. a hexagon

g. a parallelogram that is not a rectangle

h. an octagon

i. a circle

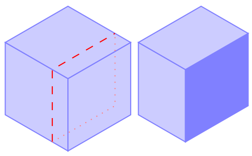
Solution. First, a comment about plane sections of a general polyhedron. Each side of a plane section comes from cutting through a face on the polyhedron. When two planes in space intersect, they intersect in a line. Thus the edges of a plane section all have to be line segments; no curved edges are possible. So the answer to part i) is "no, it is impossible to get a circle" because any plane section of a polyhedron has to be a polygon. Furthermore, since a plane section intersects each face in just one line segment, the plane section cannot have more edges than the polyhedron has faces.

Since a cube has 6 faces, a plane section can have at most 6 sides; so this answers part h in the negative: no octagons (and in fact, no seven-sided polygons either).

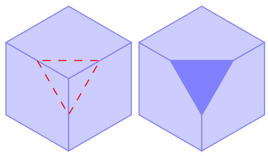
a. A square cross section can be created by slicing the cube by a plane parallel to one of its sides. This is the only way to get a square as a section of the cube; furthermore they are all of the same size as any face.

b. An equilateral triangle can be obtained by a plane section by cutting the cube by a plane that is perpendicular to the diagonal joining two opposed vertices of the cube. The largest such triangle is obtained when the plane of the section includes three vertices of the cube.

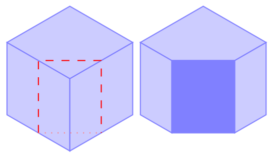
c. A plane that is perpendicular to a face, but not parallel to any face will cut the cube in a rectangle that is not a square. Every such rectangle has area less than that of a face.



a.



b.

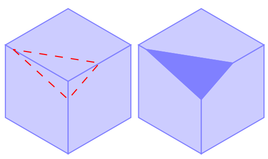


c.

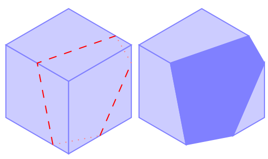
d. Pick a vertex, let's say A, and consider the three edges meeting at the vertex. Construct a plane that contains a point near a vertex (other than vertex A) on one of the three edges, a point in the middle of another one of the edges, and a third point that is neither in the middle nor coinciding with the first point. Slicing the cube with this plane creates a cross section that is a triangle, but not an equilateral triangle; it is a scalene triangle. Notice that if any two selected points are equidistant from the original vertex, the cross section would be an isosceles triangle.

e. To get a pentagon, slice with a plane going through five of the six faces of the cube.

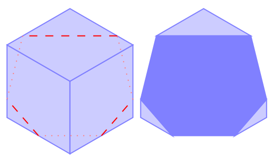
f. To get a hexagon, slice with a plane going through all six faces of the cube.



d.

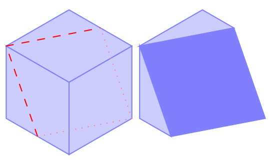


e.



f.

g. To create a non-rectangular parallelogram, slice with a plane from the top face to the bottom. The slice cannot be parallel to any side of the top face, and the slice must not be vertical. This allows the cut to form no 90° in angles. One example is to cut through the top face at a corner and a midpoint of a non-adjacent side, and cut to a different corner and midpoint in the bottom face.

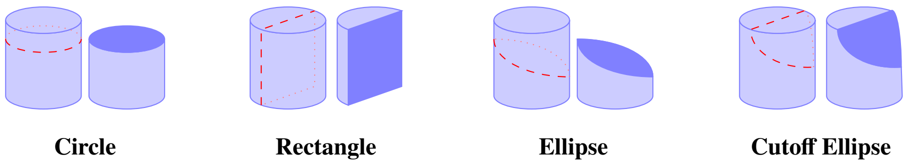


g.

Example 15.

What shapes can be created by one slice through a circular cylinder?

Solution. Possible plane sections are: a circle (cut parallel to the base), a rectangle (cut perpendicular to the base), ellipse, or a cutoff ellipse.



Example 16.

What shapes can be created by a slice through a square pyramid?

Solution. Refer to Figure 15.

a. If the pyramid is cut with a plane parallel to the base, the intersection of the pyramid and the plane is a square cross section.

b. If the pyramid is cut with a plane passing through the top vertex and perpendicular to the base, the intersection of the pyramid and the plane is a triangular cross-section.

c. If the pyramid is cut with a plane perpendicular to the base and parallel to one of the edges of the base, but not through the top vertex, the intersection of the pyramid and the plane is an isosceles trapezoidal cross-section.

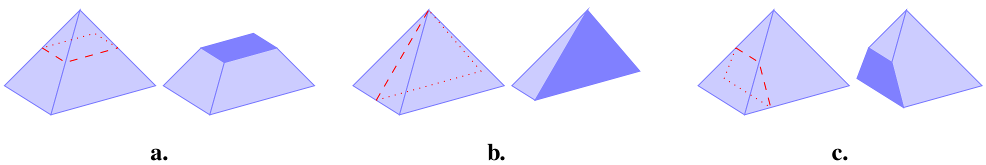


Figure 15

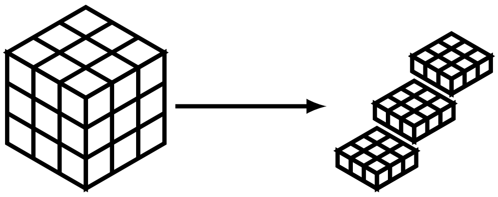
If the plane of the section is neither perpendicular to the base, nor parallel to an edge, can we find other polygons?

Volume

The volume of a solid shape is a measure of how much three-dimensional space the shape takes up. What does it mean to say that the volume of a solid shape is 27 cubic centimeters? It means that the solid shape could be made (without leaving any gaps) with a total of 27 1 cm × 1 cm × 1 cm cubes, allowing cubes to be cut apart and pieces moved if necessary. If we thought of a box as subdivided into layers, and each layer as made up of 1unit × 1unit × 1unit cubes, and each small cube has a volume 1 cubic unit, then the volume of the whole box (in cubic units) is the sum of the volumes of the cubes, which is just the number of cubes.

Consider the popular puzzle called *Rubik's Cube*.

When you think about a traditional Rubik's Cube there are three layers. In each of these layers are nine smaller cubes. When you multiply three by nine you get twenty-seven.



The most basic way to determine the volume of a solid shape is to make the shape out of cubes (filling the inside completely) and to count how many cubes it took. Now not every solid shape is made out of cubes, but if we take the unit cube small enough, this method will produce a good approximation of the volume. Although primitive, this method is important, because it relies directly on the definition of volume and therefore emphasizes the meaning of volume.

A way to find the volume of a solid shape is to understand how it can be developed out of two dimensional figures. For example, as we saw above, a cube can be viewed as a stack of squares, all of which have the same side lengths. We now use this idea to calculate volume. Let's look at prisms and cones, as they are solid shapes created by drawing out a planar polygon. We focus on sections by planes parallel to the base of the figure. If the sections are all of the same size and shape, we have a *prism* and if the they are scalings of the original we have a *cone*. Let's look at these two types more closely.

Volume of a Prism

A prism is described as the solid formed by drawing out a figure on a plane for a specified distance along parallel lines emanating from the plane. If the direction is perpendicular to the starting plane, it is called a *right prism*. We shall focus on right prisms. We want to establish this formula to compute the volume of prisms:

- The volume of a prism is the product of the height by the area of the base. That is, if the area of the base is B and the height is h, volume is $V = Bh$.

Of the three prisms illustrated in Figure 13, the first two can be viewed as having been formed by drawing out any one of its faces in the perpendicular direction. So, in calculating area we will make a choice of face to serve as base; in most cases it doesn't matter, and often the context directs us to a proper choice of base.

Example 17.

Figure 16

The National Press Building on Fourteenth Street and Avenue F is 14 stories high, with 12 feet to each story. It has 150 feet of frontage on 14th St, and 200 feet on Ave F. The building has the shape of a rectangular prism. What is its volume?

We view the building as formed by drawing a 150×200 rectangle upwards for 14 stories. Now the area of the base is $150 \times 200 = 30,000$ sq. ft. Since each story is 12 feet high, the volume of each story is $12 \times 30,000 = 360,000$ cu. ft., and as the building is made up of a stack of 14 stories identical to the first one, the total volume is $14 \times 360,000 = 5,040,000$ cubic feet.

Now, we turn to prisms that do not have a rectangular base, to see that the above assertion about volume is true. Start at the planar figure at the base of a prism: its area is approximated by covering the the figure with a grid of squares, and counting the number of squares inside the figure. The finer the grid the better the approximation. Now, Figure 17 is of a prism over a trapezoidal base B. If we cover B with a grid, we can draw out that grid along

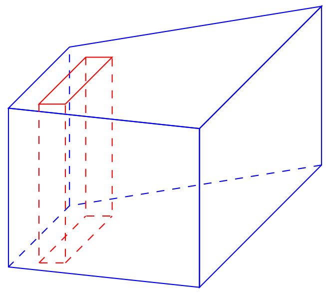


Figure 17

the parallel lines, getting a decomposition of that region into rectangular prisms, the base of each is a rectangle in the planar grid. Our figure shows a typical such rectangle. By filling the solid with rectangular prisms of this type, *all with the same height*, then adding together the volumes of all the rectangular prisms inside the prism, we get the formula "Area = base×height" for this approximation. As the grid gets finer and finer, the approximations get better and better, but the formula remains the same. So the statement $V = Bh$ is confirmed for the general prism.

Example 18.

The Pentagon, the headquarters of the U.S. Department of Defense, is a regular five-sided figure with a total of 6.5 million square feet of floor space on seven levels, two of which are underground. The side length of the interior central plaza is about 1/3 the side length of the building.

a. What is the measure of the*footprint* of the Pentagon? The footprint is the total area occupied by the building together with the central plaza.

b. What is the area of the central plaza?

c. There are 11 feet of elevation between floors of the Pentagon. What is the total volume of the above-ground building?

Solution.

a. The image shows the Pentagon to be a prism - in the sense that all floors are of the same shape and size; indeed all sections by planes parallel to the ground are of the same shape and size. Thus each floor of the building comprises 1/7 of 6.5 million sq. ft., or 928,571 sq. ft. But this is the area of the base floor of the building, not the footprint, which includes the central plaza, We are told that the the length of a side of the plaza is one-third the side length of the building. Since the plaza and the building have the the same shape, that tells us that the footprint of the plaza is a downscaling of the footprint of the entire Pentagon by a linear scale factor of 1/3. Since area scales by the square of the linear scale factor, we conclude that the area of the plaza is 1/9th of the are of the footprint. Thus the area of the floor of the building, 928,000 square feet is 8/9 of the area of the footprint. The answer then, to a) is that the area of the footprint is $\frac{9}{8}(928,000) = 1,044,000$ sq. ft.

b. The plaza is 1/9 of the footprint, so its area is $\frac{1}{9}(1,044,000) = 116,000$ sq. ft.

c. The reason this figure (the volume of the building) is interesting is to estimate the cost of heating the building in winter, and air-conditioning it in summer. So, now we are interested only in the volume of the building that is above ground. Since there are 5 stories above ground, each of height 11 feet, the bullding stands 55 feet high. The area of the base is 928,000 sq. ft., so the volume of the building above ground is $55 \times 928,000 = 51,040,000$ cu. ft.

Ifthe prism is not a right prism, as in Figure 28, can we still find its volume?

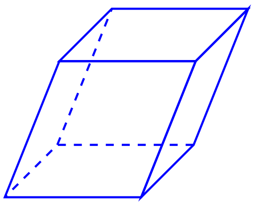


Figure 18

We can view the figure as a collection of copies of the base figure, but this time, not directly on top of one another, but each moved somewhat askew, as on the right in Figure 19:

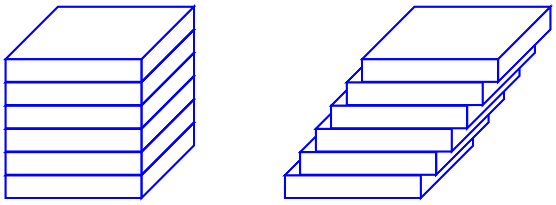


Figure 19

The volume of the stack of boxes together is the same in both figures. If we take the height of all the boxes small enough, we get a very good approximation of the volume of the figure above.

In this example, we see an application of

- **Cavalieri's principle**: Suppose that we stand two figures side by side. Suppose that every horizontal slice through the two figures gives two planar figures of the same area. Then the volume of the two solid figures is the same.

Notice that we do not require that the figures have the same size and shape, only that they have the same area. But, since figures of the same size and shape have the same area, Cavalieri's principle applies to all prisms.

Volume of a Pyramid

A pyramid has been described as the solid formed by the aggregate of line segments joining points on a given polygonal figure (the base) in a plane to a single point A (the apex).

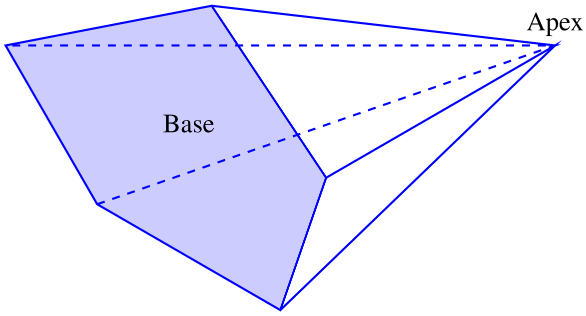


Figure 20

If the base has a center, and the line from the center to the apex is perpendicular to the base, we call the solid a *right pyramid*. The figure above is a generic pyramid whose base is a polygon with five sides. The shapes in Figure 14 are all right pyramids.

The *height* of a pyramid is the distance from the apex to the plane of the base, The formula for the volume of a pyramid is:

- The volume of a pyramid is one-third the product of the height by the area of the base. That is, if the area of the base is B and the height is h, volume is $V = \frac{1}{3}Bh$.

This fact was discovered by the the ancient Greeks by direct experimentation with a variety of pyramids. Here is a statement of their conclusion: Start with a pyramid whose apex lies somewhere over the base. The *circumscribed prism* is the prism of the same height with the same base. Figure 21 depicts this pair for a right pyramid over a rectangle.

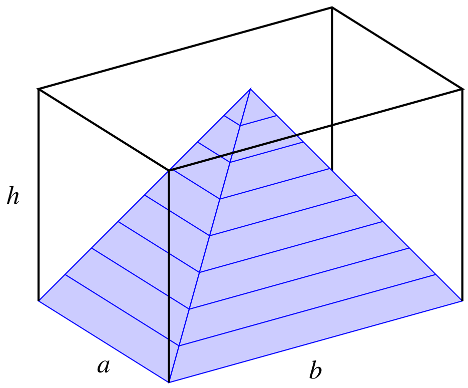


Figure 21

What was observed was that the volume of the pyramid is 1/3 that of its its circumscribed prism. This can be shown by creating containers with these shapes and comparing the volume of water that fills the objects. It takes exactly three fillings of the pyramid to fill its circumscribed prism. The Greeks went further with a specific pyramid, the tetrahedron. Three models of the tetrahedron can be put together to completely fill the circuscribed cube.

By extension, the Greeks concluded that the volume of a circular cone is 1/3 the volume of its circumscribed cylinder (students will return to this in 8th grade.

Still, this is not a geometric argument, as the ancient Greeks would have wanted it to be. Even today, we have no satisfactory way to visualize putting three circular cones of the same size and shape into the circumscribed cylinder.

Here is a construction that comes close. Use Figure 22 to follow along. Consider the cone whose base is a square, and whose height is equal to the side length of the base. Furthermore, put the apex of the cone directly over one of the vertices of the base square. Construct the cube that is its circumscribed prism .

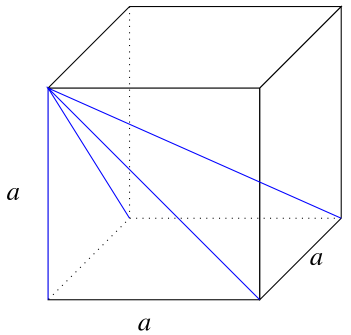
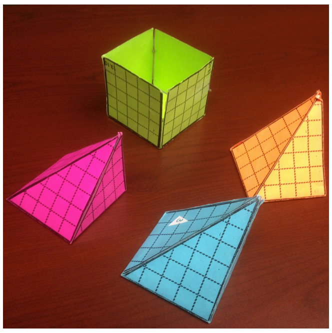
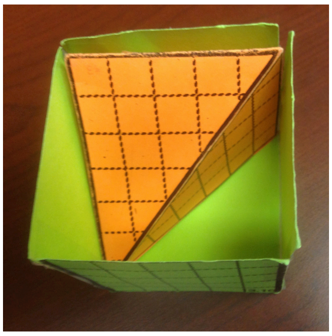
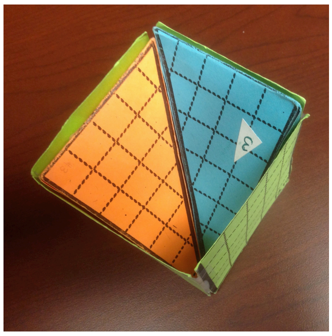
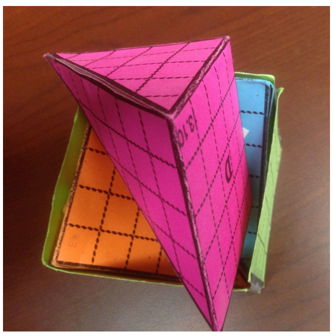


Figure 22

Make a box of the dimensions of circumscribed cube, and a pyramid as indicated by the figure. The following sequence of figures illustrate how to fill the cube with three copies of the cone.




Now, this construction, and the physical measurements of volume with water all support the conclusion that the volume of a cone is 1/3 the volume of its circumscribed prism. But, for the Greeks, this did not explain why? Why 1/3? When we move from a right triangle to its circumscribed rectangle, we have a factor of 1/2; now, moving from two dimensions to three, we introduce a factor of 1/3. This seemingly strange geometric fact, together with the unconstructive nature of the parallel postulate, worried mathematicians strictly adherent to the principle of logic for almost 2 thousand years.Today the factor 1/3 is easily understood thanks to a calculation by Cavalieri of an area bounded by parabola, and later incorporated as one of the building stones of the Calculus.

Example 19.

> The 7th graders at Albert R. Lyman Middle School were helping to renovate a playground for the kindergartners at the nearby Blanding Elementary School. Blanding City regulations require that the sand underneath the swings be at least 15 inches deep. The sand under both swing sets was only 12 inches deep when they started. The rectangular area under the small swing set measures 9 feet by 12 feet and required 40 bags of sand to increase the depth of 3 inches. How many bags of sand will the students need to cover the rectangular area under the large swing set if it is 1.5 times as long and 1.5 times as wide as the area under the small swing set?
>
> Solution. There are many different ways to approach and solve this problem. Let's consider three approaches that student's might take with respect to volume, scale factor or unit rate.
>
> Solution 1 (volume): 3 inches is 1/4=0.25 feet, so the volume of sand that was used is $0.25\times9\times12 = 27$ cubic feet. The amount of sand needed for an area that is 1.5 times as long and 1.5 times as wide would be $0.25 \times (1.5?9) \times (1.5 \times 12) = 60.75$ cubic feet. We know that 40 bags covers 27 cubic feet. Since the amount of sand for the large swing set is $60.75 \div 27 = 2.25$ times as large, they will need 2.25 times as many bags. Since $2.25 \times 40 = 90$, they will need 90 bags of sand for the large swing set.
>
> Solution 2 (scale factor): Since we have to multiply both the length and the width by 1.5, the area that needs to be covered is $1.5^2 = 2.25$ times as large. Since the depth of sand is the same, the amount of sand needed for the large swing set is 2.25 times as much as is needed for the small swing set, and they will need 2.25 times as many bags. Since $2.25 \times 40 = 90$, they will needs 90 bags of sand for the large swing set.
>
> Solution 3 (unit rate): The area covered under the small swing set is $9 \times 12 = 108$ square feet. Since the depth is the same everywhere, and we know that 40 bags covers 108 square feet, they can cover $108 \div 40 = 2.7$ square feet per bag. The area they need to cover under the large swing set is $1.5^2 = 2.25$ times as big as the area under the small swing set, which is $2.25 \times 108 = 243$ square feet. If we divide the number of square feet we need to cover by the area covered per bag, we will get the total number of bags we need; $243 \div 2.7 = 90$. So they will need 90 bags of sand for the large swing set.

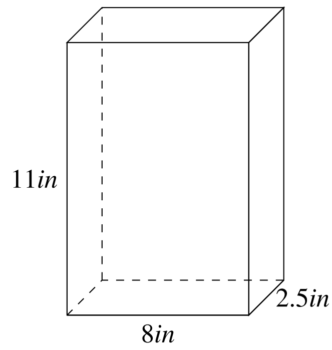
Contrasting and Relating Volume and Surface Area

In sixth grade, using the idea of nets, students worked out strategies to find the area of the surface of a polygonal figure in space: adding up the surface areas of all the faces. The decomposition into nets provided a way to organize this computation. So, in particular, a rectangular prism of side lengths 2, 3 and 5 units has six faces, 2 each of dimensions 2×3 sq. un., 2×5 sq. un., and 3×5 sq. un., so the surface area is $2(2\times3+2\times5+3\times5) = 2(6+10+15) = 62$ sq. un. Here we want to understand the difference between volume and surface area and the relation between them.

First of all, why is surface area important? A painter would be interested in the surface area of a room, rather than volume. Chemotherapy treatment of cancer takes place through the surface of the growth, so the surface area of the cancer is a more important parameter than its volume.

Example 20.

> Manufacturers sell breakfast cereals by volume or weight (usually 25 oz.) but determine the size of containers on economic and aesthetic grounds. Here is a typical example. What is the area of the material necessary to cover the surface of the box.



SOLUTION. A rectangular prism has 6 faces, identical in opposing pairs. The dimensions of the faces, in this case, are 11×8, 11×2.5, and 8×2.5 sq. in. Doing the multiplication we have two faces of 88 sq. in., 2 faces of 27.5 sq. in. and 2 faces of 20 sq. in. Therefore the total area is 32(88+27.5+20) = 135.5 sq. in.

EXAMPLE 21.

In the movie *Despicable Me*, an inflatable model of The Great Pyramid of Giza in Egypt was created by Vector to trick people into thinking that the actual pyramid had not been stolen. When inflated, the false Great Pyramid was 225 m high, with a slant height of 230.5 m for any one of the triangle faces (by *slant* height, we mean the distance from its base to the apex of the pyramid in the plane of the triangle. The base a square with each side 100 m in length. How much material did Vector need in order to re-create The Great Pyramid of Giza?

SOLUTION.

$$\text{Area}_{triangle} = \frac{1}{2}(100 \times 230.5) = 11,525 \text{ sq. m}$$

$$\text{Area}_{base} = (100 \times 100) = 10,000 \text{ sq. m}$$

$$\text{Total Surface Area} = 4(10,000) + 11,525 = 51,52 \text{ sq. m.}$$

EXAMPLE 22.

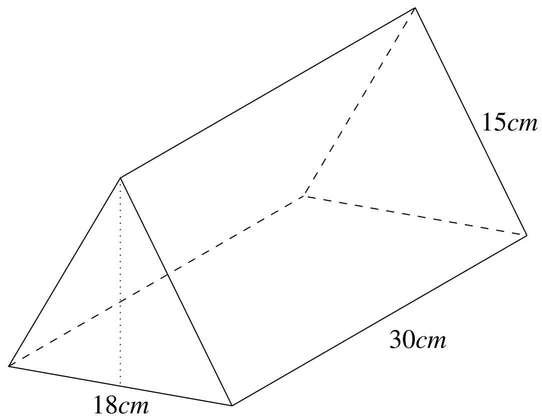
A baker creates fantastic cupcakes that can each be comfortably enclosed in a cube of side length 2 in and square base. She wants to deliver these cupcakes to stores in batches of 75 in a large box that has 3 layers of cupcakes. What is the volume of the box, and what is its surface area?

SOLUTION. Since the 75 cupcakes are placed in the box in three square layers, each layer has 25 cupcakes in a 5×5 array. Each cupcake takes up a space that is $2 \times 2 \times 2$ inches, so the box has a base that is 10×10 sq. in. and a height of 6 in. The volume of the box is the product of its side length: $V = 10 \times 10 \times 6 = 600$ cu. in. The area of the bottom is 100 sq. in. and the area of a side is 60 sq. in. Since the top has the same area as the bottom, and there are 4 sides, the surface area is $A = 2(100)+4(60)$ = 440 sq. in.

In packaging, one must know the edge dimensions, the surface area and the volume; however, some of these are more significant that others. For example, if we are packaging heavy objects, the volume is most important,but if we are wrapping it with an expensive paper, surface area may be paramount, and if we tie it up in gold ribbon, the edge dimensions count the most.

EXAMPLE 23.

Mr. Brewer purchased a box of his favorite chocolates. The box is in the shape of a prism whose base is an isosceles triangule (see the diagram below), and the edge dimensions of the prism are as shown. If the volume of the box is 3,240 cubic centimeters, what is the height of the triangular face of the box? How much packaging material was used to construct the package



Solution. Volume is found by multiplying the area of the base (isosceles triangle) by the length of the prism: $V = Bl$. Here we are given the dimensions $V = 3240$ cu. cm., $l = 30$ cm., so we must have $B = 108$ sq. cm. Now the area of a triangle is one-half the product of the base and the height, and here we know that the base is 18 cm, so we have $\frac{1}{2}(18)h = 108$, giving us $h = 216/18 = 12$ cm.

The problem also asks for the surface area of the package. Find the area of each face and add:

2 triangular bases, each of which is 108 sq. cm. contribute 216 sq. cm;

2 rectangular faces (the side faces in the diagram) that are 15×30 each contribute 450 sq. cm.;

One 30 cm. $\times$ 18 rcm. rectangular bottom contributes 540 sq. cm.

The total produces $216 + 2 \times 450 + 540 = 1656$ sq. cm.

Is there a relationship between surface area and volume? Can rectangular prisms with different dimensions have the same volume? Do rectangular prisms with same volume have the same surface area?

Example 24.

For shipping purposes, cubes of fudge need to be packaged in boxes that are rectangular prisms. Knowing the company only sells their fudge cubes in groups of 24, what are the possible dimensions for the boxes?

Solution. . Begin by recording the information in table form.

Length	Width	Height	Volume	Surface Area
1	1	24	24 cu. cm	98 sq. cm
2	1	12	24 cu. cm	76 sq. cm
3	1	8	24 cu. cm	70 sq. cm
4	1	6	24 cu. cm	68 sq. cm
2	2	6	24 cu. cm	56 sq. cm
2	3	4	24 cu. cm	52 sq. cm

Which of the packages requires the least and the greatest amount of material and why would it be important? The package that requires the least amount of material is the $2 \times 3 \times 4$ package and the package that requires the greatest amount of material is the $1 \times 1 \times 24$. Why is the amount of material important to a company? Because material costs money and the more material would equate to a higher cost.

What conclusion can we make about the shape of the package with the smallest and greatest surface area and what would you recommend to the fudge company? Notice that the shape of the package with the smallest surface area is the package that most closely resembles a cube and the package with the greatest surface area is the package that is least like a cube. Recommending the $2 \times 3 \times 4$ package would mean that the company would save money by using the least amount of material possible.

Looking at the table, what is the relationship between surface area and volume? Notice that surface area decreases as the rectangular prisms move closer to the shape of the cube. Another key revelation is that rectangular prisms with different dimensions can have the same volume. Lastly, rectangular prisms with the same volume can have different surface area. For example, the number of exposed faces of each unit cube is different for each prism. The $1 \times 1 \times 24$ prism has 22 cubes with four exposed faces and the two end cubes have five exposed faces. The $2 \times 3 \times 4$ has 8 cubes with three exposed faces (the corners), 12 cubes with two exposed faces, and 4 cubes with one exposed face.

Example 25.

I was planning to send a gadget that is 8 in. by 12 in. by 14 in. to my Aunt Sarah.

a. What is the volume of the box I would need?

b. As it turns out, that gadget is no longer available, but I can send her a scale-reduced model of the gadget, reduced to 60% in every linear dimension. Now what volume box do I need?

c. Looking further on the internet I find an opportunity to purchase and send a similar object that is 4 in longer than the one I was considering. However, I am not sure which dimension is the length (Note, length is typically the longest dimension).What are the possible volumes of these packages? What are the possible surface areas of these packages?

Solution.

a. The volume is $8 \times 12 \times 14 = 1344$ cu. in. Now, a cubic foot is the same as 1728 cu. in, so the package contains $1344/1728 = .78$ cu.in.

b. This reduced-size gadget has been reduced by 40% to 60% of its original value in every linear dimension. Thus the new volume is $(0, 60)^3 \times 0.78 = 0.16848$ cu.ft., or about a fifth of a cubic foot.

c. The three possibilities are: first, $(8 + 4) \times 12 \times 14 = 2016$ cu ft; second, $8 \times 16 \times 14 = 1792$ cu. ft.: and the third is $8 \times 12 \times 18 = 1728$ cu.in.

Example 26.

For a rectangular prism, if we know the area of each its faces, do we know its volume?

Solution. The answer is "yes," and there are many ways to show this. First, we should suspect that the answer is "yes" since it takes 3 numbers (the lengths of the sides) to determine the volume, and if we are given the areas of the faces, we have 3 numbers, so that should suffice. This suggests that, when we are given the face areas, we can solve for the edge lengths.This is in fact true: Let a, b, c be the side lengths, and A, B, C the given face areas. We can relate these areas to the side lengths by the equations:

$$A = bc \quad B = ac \quad C = ab\ .$$

If we multiply the left sides together we get ABC, and if we multiply the right sides together we get $a^2b^2c^2 = V^2$, since the volume $V = abc$. Thus V is the square root of the produt of the sides.