Chapter 1
Linear Equations in One Variable

The first three chapters of grade 8 form a unit that completes the discussion of linear equations started in 6th grade, and their solution by graphical and algebraic techniques. The emphasis during these three chapters moves gradually from that of "equations" and "unknowns" to that of "functions" and "variables." The first chapter is about solving linear equations; in the second we move to the graphical interpretation of linear expressions and the understanding of the constant rate of change for linear functions (to be compared to the constant of proportionality of a proportional relation), and then to the "slope" of a line, and how it can be interpreted as the rate of change of the associated linear expression. Note that in this discussion we have been using the word *expression* rather than *function*, so that the students will become familiar with the idea of evaluating linear expressions, and graphing those values, as a lead in to the concept of function. This is engaged in the third chapter in which the entire subject of linear functions is brought together and examined from a variety of perspectives. In Chaprer 4 we return to solutions of equations, this time with pairs of equations in two variables for which we seek values of the variables that solve both equations (the solutions of the simultaneous equations). Finally, in Chapter 5 our focus turns completely to the concept of function, shifting emphasis to describing the relation between the two variables, rather than the mechanics of the function.

This chapter begins with a focus on the distinction between expressions and equations. The analogy is with language: the analog of "sentence" is *equation* and that of "phrase" is *expression*. An equation is a specific kind of sentence: it expresses the equality between two expressions. These equations involve certain specific numbers and letters. We refer to the letters as *unknowns* - that is they represent actual numbers which are not yet made specific; indeed, the task is to find the values of the unknowns that make the equation true. If an equation is true for all possible numerical values of the unknowns (such as $x + x = 2x$), then the equation is said to be an *equivalence*. Arithmetic operations transform expressions into equivalent expressions; we come to understand that for linear expressions, the converse is true: we can get from one expression to an equivalent one by a sequence of arithmetic operations.

In chapter 2 we begin to change the way we look at the letters used in algebra from that of *unknown* to *variable*, and together with that, the understanding of an equation involving two variables as expressing a relation between them. We do this first in the context of proportion, but then go to general linear equations and the ideas of *rate of change* and *slope* of the graph. But in chapter 1 we are only interested in finding that (or those) number(s), if any, which when substituted for the unknown make the equation true. These are the *solutions*. We do this in specific contexts, seeing how to translate language sentences about numbers into equations involving linear expressions.

Linear Expressions

A *linear expression* is a formula consisting of a sum of terms of the form ax and b, where a and b are numbers and x represents an unknown. By *unknown* we mean a symbol which stands for a number; it could be a specific one yet to be determined, or one yet to be chosen, or any possible number, depending upon the context. Since "x" represents an unknown, we could replace it by any letter and still have the same sentence. In particular, in solving a problem in context, it is a good idea to pick a letter suggested by the context.

Example 1.

These are all linear expressions:

a. $3x - 5$ **b.** $3t$ **c.** -5 **d.** $3u + 2u + 17$ **e.** $3x - \frac{12x - 16}{4}$ **f.** $\frac{2}{5}y + \frac{3}{10}$ **g.** $6(2x - 5) + 11$

Notice (as in examples **b.** and **c.**) that we could have either a or b, or both, equal to zero. When we have a particular problem, we are unlikely to start with a linear expression of the form **c**, but, in our manipulations with the expression, we may end up there. For example, when we combine terms in **e** we end up with $3x - 3x + 4$, which is just 4.

To *evaluate* a linear expression is to substitute a number for the unknown and calculate the resulting value. For example, if we evaluate **d** at $u = 1$, we get 22, at $u = -3.5$, we get -0.5. Often we are interested in how the value of the expression changes as we change the value of x and evaluate, and so we generate a table of corresponding values of x and the expression, and go on to graph those points on a coordinate plane. Were we to do so, it always turns out that the points lie on a straight line, and that is why expressions of this form are called "linear". But we are getting ahead of ourselves; we will return to this discussion in the next chapter.

It is often the case that different linear expressions have the same meaning: for example $x + x$ and $2x$ have the same meaning, as do $x - x$ and 0. By the *same meaning* we mean that a substitution of any number for the unknown x in each expression produces the same numerical result.

Example 2.

Consider these linear expressions:

a. $2(x + 5)$ **b.** $2x + 10$ **c.** $x + 10$ **d.** $2x + 5$

a. and **b.** have the same meaning, since every substitution of a number for x gives the same result. As for **a.** and **c.** , although the substitution of 0 for x produces the same result, it will not work for any other number. And the situation for **a.** and **d.** is even worse: there is no number for which they both give the same result,

How do we know that **a.** and **b.** have the same meaning, since we cannot text *every* number? The answer lies in the laws of arithmetic. We say that two linear expressions are *equivalent* if we can move from one expression to the other using the laws of arithmetic. When two linear expressions are equivalent, they have the same meaning: to be precise, any number substituted for the unknown in the expression always returns the same value. Indeed, this is why the "laws of arithmetic" are called laws: they preserve the meaning of the expression. This is very convenient: to show that two expressions have the same meaning, we do not have to check every number (an impossible task in any event); it is enough to show that we can move from one expression to the other using the laws of arithmetic. On the other hand, to show that two expressions are not equivalent, just find one number that gives different results when substituted for the unknown in the expressions.

In this chapter we want to work precisely with these ideas of equivalence of linear expression, leading to *simplifying* and *solving*. In the next chapter we will see that the graph of a linear expression (where y is the value of the expression for a value of x) is a straight line. Since a line is determined by any two points on the line, we will see that two linear expressions are equivalent, if they return the same values for any numbers substituting for the unknown.

Every linear expression is equivalent to one in the form $ax + b$ by applying the laws of arithmetic. This process is sometimes called *simplifying* the expression. However, "simplify" is not always what we want to do with an expression. For example, if we change $7x - 28$ to $7(x - 4)$, then we learn that in the equation $y = 7x - 28$, y is proportional to $x - 4$. So, if we are interested in the behavior of x and y in the equation $y = 7x - 28$, the form $y = 7(x - 4)$ is "simpler." In general, the word "simplify" should be tied to the goal of the work being done; right here it is to minimize the number of symbols necessary to understand the expression.

Every one of the above expressions in example 1 can be put in the form $ax + b$ for some numbers a and b, using arithmetic operations. As for examples **a**, **b**, **c** and **f**, they are already in that form. Let's now look at the others, and a few more just to review all the possibilities.

Example 3.

- **a.** example 1d: $3u + 2u + 17$ is equivalent to $5u + 17$. For we can combine similar terms: "three u's plus two u's" is the same as "$5u$."
- **b.** example 1e: By doing the division implied by the fraction, we see that $3x - \frac{12x-16}{4}$ is equivalent to $3x - 3x + 4$, which is just 4 (as mentioned above).
- **c.** example 1g: Distribute the 6 in $6(2x - 5) + 11$ to get $12x - 30 + 11$, and now add $-30 + 11$, to get $12x - 19$.
- **d.** $4x + 5$ and $2x + 3x + 5$ are not equivalent. Substitute 1 for x, and obtain 9 in the first expression and 10 in the second.
- **e.** $3x + 5$ and $6x - 1$ are not equivalent: if we substitute 1 for x, we get 8 in the first expression and 5 in the second. But be careful: if we substitute 2 for x, we get the same result: $3(2) + 5 = 11$ and $6(2) - 1 = 11$. Since we can find at least one value for the unknown that gives different results to the expression, they are not equivalent.

Example 4.

$6x - 20 + 2(x - 4)$ and $4(2x - 7)$ are equivalent.

Let's go through the steps, giving clear reference to the relevant laws of arithmetic.

Step 1. Start with $6x - 20 + 2(x - 4)$. Distribute the 2 to remove the parentheses, to get: $6x - 20 + 2x - 8$.

Step 2. Combine like terms to get: $8x - 28$

Step 3. Factor out 4 to get $4(2x - 7)$ which is precisely the second expression.

There are many ways to go from one expression to an equivalent one. For example, we could distribute and collect terms in both expressions to obtain $8x - 28$ from each. To put this another way: two expressions are equivalent if they are both equivalent to a third expression.

In summary, the end result of simplification is an expression of the form $ax + b$: This is always the case: any linear expression simplifies to the form $ax + b$. The word "simplify" is often ambiguous - it usually depends upon where it is you want to go with the expression; in this case, it is to the form $ax + b$.

Section 1.1. Solving linear equations: obtaining the desired value of an expression

Solve linear equations with rational number coefficients, including equations whose solutions require expanding expressions using the distributive property and collecting like terms. 8.EE.7.ab

In the introduction to this chapter, we talked about "evaluating expressions". Here we ask: given a linear expression, and a number c, for what value of the unknown does the expression compute to c? This can be restated as:

given the expression $ax + b$ and a number c, find the value of x that produces that c. Let's first review what was done in grade 7.

Example 5.

a. For what x does $2x + 5$ evaluate to 17? Otherwise put: solve $2x + 5 = 17$.

Solution. Subtract 5 from both sides of the equation to get $2x = 17 - 5$. Replace $17 - 5$ by 12 to get $2x = 12$. Now divide both sides by 2 to get $x = 6$.

b. For what x does $2(x + 5)$ evaluate to 24? Otherwise put: solve $2(x - 5) = 24$.

Solution. Divide both sides by 2 to get $x - 5 = 12$. Now add 5 to both sides: $x = 17$.

Note that in the second problem, we'd rather not use the distributive property: it is easier and quicker to first divide by 2 than to distribute the 2.

Now we want to work more complicated expressions. The procedure will be the same, except that first we have to appropriately simplify the expression. Let's work with the expressions **e**, **f** and **g** of example 1.

Example 6.

a. For what value or x is $3x - \frac{12x - 16}{4}$ equal to 5?

Solution. First, we reduce the fraction to obtain the equation $3x - (3x - 4) = 5$, and then use the distributive property to get $3x - 3x + 4 = 5$. Then combine terms to obtain $4 = 5$. Since 4 is not equal to 5, there is no value of x to obtain 5 from this expression.

b. Let's slightly change the expression so as to obtain a more satisfying result. For what value or x is $5x - \frac{12x - 16}{4}$ equal to 25?

Solution. Again, we reduce the fraction, this time obtaining $5x - 3x + 4 = 25$. Combining terms, this becomes $2x + 4 = 25$, which has the solution $x = \frac{21}{2}$.

Example 7.

For what value of x is $\frac{2}{5}x + \frac{3}{10}$ equal to 0.375?

Solution. Otherwise put, solve

$$\frac{2}{5}x + \frac{3}{10} = 0.375$$

First, Multiply both sides by 10 to obtain

$$4x + 3 = 3.75$$

Subtract 3 from both sides getting $4x = 0.75$, and divide by 4 getting $x = 0.1875$

This is a good time to point out that there can be many ways to solve a problem, and in this case, there may be better ways. Noticing that the notation is hybrid (we have both fractions and decimals) we could move to one notation or the other.

Yet another way would be to write all numbers as decimals to get $0.4x + 0.3 = 0.375$ and now multiply by 10 to get $4x + 3 = 3.75$, and now proceed as above.

Write all numbers as fractions to get

$$\frac{2}{5}x + \frac{3}{10} = \frac{3}{8}$$

Multiply by 40 to eliminate denominators, getting

$$16x + 12 = 15$$

Subtract 12 from both sides to get $x = \frac{3}{16}$

Example 8.

a. For what value of x is $6(2x - 5) + 11 = 53$?
b. For what value of x is $6(2x - 5) + 11 = y$?

Solution. **a.** We want to illustrate two different ways to solve this problem.

a1. Distribute the 6 and add −30 to 11: $12x - 19 = 53$,

Add 19 to both sides: $12x = 72$, and now divide by 12 to get $x = 6$

a2. Subtract 11 from both sides : $6(2x - 5) = 42$,

Divide by 6: $2x - 5 = 7$

Add 5 to both sides: $2x = 12$,

and now divide by 2 to get the answer $x = 6$.

b. First, apply the first two steps of **a1** to get $12x = y + 19$. Now divide by 12 to get the result

$$x = \frac{y + 19}{12}$$

Section 1.2. Solving linear equations: equating two expressions

A *linear equation* is an assertion that two linear expressions are equal. In the above, we have considered the case where one of the expressions is simply a number, and put it in the context of evaluation of expressions. Now we want to find out for what value of the unknown two expressions produce the same result. This may seem more difficult to the students, but the ideas are precisely the same. The difficulty may be this: it is clear that we can subtract 5 from both sides of the equation to get an equivalent equation, but since we don't know what x is, is it really all right to subtract $5x$ from both sides? Of course it is, since x does represent a specific number, and so the laws of arithmetic apply. Later, when we move from the concept of "unknown" to that of "variable", then x is a quite different object, representing not some particular number that we don't know just yet, but any possible number. Nevertheless, the same reasoning applies: the laws of arithmetic actually do hold for any numbers and any expressions.

If a linear equation is an assertion that two linear expressions are equal, then "solving" the equation is to find out for what numbers the assertion is true. Two linear equations are equivalent if one can be obtained from the other by a succession of applications of laws of arithmetic. The goal of solving the equation is to find a sequence of equivalent equations, starting with the given equation and ending up with something like $x = 5$. Of course, we may not end up there: just as the expression $x - x + 1$ is equivalent to the expression 1, the equation $x - x + 1 = 2$ is equivalent to $1 = 2$, which of course is false. Since it is false, no matter what value x takes there is no solution.

In general, the result of this process may be "all numbers" or "a particular number" or "no numbers". Let's look at some examples:

Example 9.

a. $2(x - 5) = 3x - 1$ **b.** $2(x + 5) = 2x + 10$ **c.** $7 = 5$ **d.** $7x = 5x$

e. $3(x - 5) = 2x$ **f.** $7 = 5 + 2$ **g.** $7x = 7x + 1$

The examples presented here are designed to indicate the breadth of issues that may come up as students learn this subject, and not to provide instructions. As you look through them keep in mind that the truth or falsity of the equation is something to be determined: it is our task. This is different from the validity of the equation as a statement. To illustrate: "Julius Caesar was the first President of the United States" is a valid statement, but false. The assertion "Mr. XXX was the first President of the United States" is a valid statement, but doesn't tell us much (except that that person was male). The equally valid and true statement is "George Washington was the first President of the United States'." Analogously, in example 8, **a.** is true for one value of x, **b**. for all values of x and **c.** for no values of x. If the equation is true for the substitution of every number for the unknown, it is an equivalence. Now, the reader might conclude that we cannot substitute *every* number for the unknown, so we can never be sure it is an equivalence. However, for linear equations, after we verify in the next chapter that the graph is a straight line, it follows from the fact that a straight line is determined by just two points, that we need only check two values of x. Now, if an equation is not an equivalence, it still may be true for some substitutions of x (these are called the solutions), or there may be no substitution to make it a true statement.

There are various techniques for solving a linear equation; all techniques amount to applying arithmetic operations to the equation that do not change the set of solutions. There are three kinds of operations:

1. Apply the laws of algebra to simplify the expressions; in particular, distribute to remove parentheses and combine like terms.

Transform the equation $2x + 3x = 5 + 20$ to the equation $5x = 25$. Transform the equation $6(x - 2) = 11$ to the equation $6x - 12 = 11$.

2. Add or subtract the same expression to both sides of the equation.

Transform $3x = 2 - x$ to $4x = 2$ by adding x to both sides.

3. Multiply or divide both sides of an equation by a nonzero number.

Transform the equation $2x = 8$ to $x = 4$ by dividing both sides of the equation by 2. Transform $3x = 6x - 18$ by first dividing by 3 to get $x = 2x - 6$, and then combine like terms to find $x = 6$.

These operations all transform any equation into another with the same set of solutions. What is most important is that they are effective: they succeed in solving any linear equation. Let's apply these ideas to the equations of parts **a** through **g** of example 8.

Example 8 Solutions.

a. $2(x + 5) = 3x - 1$;

Simplify the left side: $2x + 10 = 3x - 1$;

Subtract $2x$ from both sides: $10 = x - 1$;

Add 1 to both sides: $11 = x$. Thus there is one solution: $x = 11$.

b. $2(x + 5) = 2x + 10$;

Simplify the left side: $2x + 10 = 2x + 10$. Since both sides are the same expression this is true for all values of x; that is, the expression on both sides of the equals sign in b) are equivalent. Consequently, every number is a solution to this equation.

c. $7 = 5$ this is false: If we think of this as $7 + 0x = 5 + 0x$, we can assert that there is no value of x to make it true.

d. $7x = 5x$: Subtract $5x$ from both sides: $2x = 0$. Divide both sides by 2: $x = 0$, so 0 is the only solution.

e. $3(x - 5) = 2x$;

Simplify the left side: $3x - 10 = 2x$;

Subtract $2x$ from both sides: $x - 10 = 0$;

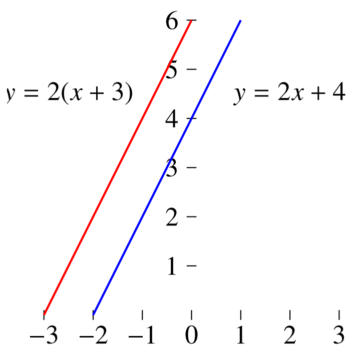
Add 10 to both sides: $x = 10$.b Therefore, there is one solution: $x = 10$.

f. $7 = 5 + 2$: combine like terms on the right: $7 = 7$. This is a true statement.

g. $7x = 7x + 1$: Subtract $7x$ from both sides: $0 = 1$. This is a false statement, so there is no solution to the original statement: no choice of value for x will make it true..

Let us take a moment to notice the exceptional cases: **b**, **c** and **g**, where we do not get a solution, but either get an equivalence (all numbers solve the equation) or there is no solution. These occur when the coefficient of x is the same on both sides of the equation. We see this in **b** after distributing the 2, and in **c** and **g** the equation starts that way.

There is a geometric representation of this situation that helps make this clear. Suppose that we start with the expressions $2x+4$ and $2(x+3)$. Let's graph the two expressions: that is, graph the equations $y = 2x+4$ and $y = 2(x+3)$.



In general, when we graph two linear expressions, we get two lines, and the point of intersection gives the value of x for which the two expressions give the same number. However in this case (see the figure), the lines are parallel, so there is no point of intersection; explaining why there is no solution. Now, if we had started out with the expression $2x + 4 = 2(x + 2)$, then the figure would have shown just one line, since the two expressions are equivalent.

An important feature of the allowable operations on equations is that they can be reversed: if an operation takes one equation to another, it can be undone:, meaning that there is an operation on the second equation that produces the first.

Example 10.

Solve $-3x + 8 = 20 + x$.

Step 1: Subtract x from both sides to get: $-4x + 8 = 20$.

Step 2: Divide both sides by -4 to get $x - 2 = -5$.

Step 3: Add 2 from both sides to get $x = -3$.

Now let's reverse the process. Start with $x = -3$.

Step 1: Subtract 2 from both sides to get $x - 2 = -5$.

Step 2: Multiply both sides by -4 to get $-4x + 8 = -20$.

Step 3: Add x to both sides to get $-3x + 8 = -20 + x$.

Summary: To solve any linear equation, use these rules, not necessarily in the order listed. Practice develops a sense of the sequence that best leads to the solution:

1. Use the distributive law to remove parentheses.
2. Combine like terms so that each side of the equation is of the form $ax + b$.
3. Add the same expression to both sides of the equation so that x appears on only one side of the equation.
4. Divide by the nonzero coefficient of x; resulting in an equation of the form $x = c$.

Example 11.

Students will need to develop a facility for discovering mistakes in the procedure, when, after checking, it is discovered that the arrived at answer does not solve the original equation. In the following determine whether or not the following arguments are correct, and if incorrect, explain the error,

a. $2(x + 5) = 13$	**b.** $3x - 15 = 24$	**c.** $2x + 3 = x + 10$
$2x + 5 = 13$	$3(x - 5) = 24$	$2\left(x + \frac{3}{2}\right) = x + 10$
$2x = 8$	$x - 5 = \frac{24}{3}$	$x + \frac{3}{2} = \frac{x}{2} + 10$
$x = 4$	$x - 5 = 8$	$\frac{x}{2} = 10 - \frac{3}{2}$
	$x = 13$	$x = 10 - 3 = 7$

Solution.

a. There is a mistake in the first step: the 2 is improperly distributed; the second line should be $2x+10 = 13$. Now subtracting 10 from both sides gives $2x = 3$ which leads to the answer $x = 3$. Now check:

$$2\left(\frac{3}{2} + 5\right) = 2 \times \left(\frac{3}{2}\right) + 2 \times 5 = 3 + 10 = 13$$

b. This is not so straightforward as the preceding problem, so first we check whether or not the answer satisfies the first equation: substitute 13 for x to get $3(13) - 15$ which is $39 - 15$, which simplifies to 24. Since this is the same as the right hand side of the equation, the answer is correct. But that does not mean that the argument is correct: we have to still check that the step from one line to the next is a correct application of arithmetic. In this case, it checks out: line one to two is the distributive property; line two to three: both sides are divided by 3; line three to four, replaces $24/3$ by the equivalent number 8 and finally we get to the last line by adding 5 to both sides.

c. First, let's check that 7 solves the first equation: with this substitution for x, the left hand side is $2(7) + 3 = 17$, and the right hand side is $7 + 10 = 17$. Now, let's check that the reasoning is correct. Well, there is a mistake in going from line two to three: we are dividing both sides by 2, but on the right, the second term (10) was not divided by two. The next step is correct: we have subtracted $x/2$ and $3/2$ from both sides. However, the step to the last line is faulty: in multiplying both sides by 2, we failed to multiply the term 10 by 2. This error has the effect of correcting the preceding error, so when we end up with $x = 7$, we accidentally ended up with the correct answer even though the steps were flawed.

Section 1.3. Creating and Solving Linear Equations to Model Real World Problems

Solve real world problems with one variable linear equations. 8.EE.7c

Up to this point in this chapter we have been discussing the questions: what does an equation tell us, "What does an equation tell us?", "What does it mean to solve an equation?", and "What is the process for solving linear equations?" However our students are most unlikely to ever see an equation to solve except in other math classes that they will take or, possibly, will teach. So why are we teaching this material? The answer is this: they will have, as professionals, workers, and as human beings living in the 21st Century, many problems to solve on a daily basis. They will need tools to think through those problems in a way that leads to an acceptable course of action. If the course of action depends upon the values of specific quantities, it is highly likely that algebra - or more often, algebraic thinking - will be the tool to apply.

It is coherent and effective algebraic thinking that is the goal of the study that started in sixth grade and continues through to high school graduation. It is *not* the protocol, or technique: we have computers and calculators to do that work. So then, the question again arises: why do we teach the protocol for solving equations? One might just as well ask: "Why does the plumber have to know his wrenches?", "Why does the dentist have to understand the dentist chair?", "Why does a carpenter have to know what a hammer can and cannot do?", "Why do we have to learn about our car before we can drive it?" The answers are so obvious that these questions hardly ever come up. But the connection between algebraic thinking and solving of real-life problems is not easily taken for granted, and one must be able to explain it - the teacher in order to scaffold the teaching, and the student in order to be motivated to learn.

Although this is the third section on linear equations, it is the most important one: the content here is the relationship between a problem stated verbally, and its algebraic restatement. This we call *mathematical modeling* of a problem. The state of one's financial resources may suggest that, while shopping, one take into consideration the relationship between cost and value. This may involve gathering of data (more shopping) and then analysis of those data. All of these operations involve algebraic thinking.

Modeling of a problem makes it possible to do calculations and make predictions. For example, we record the passage of time with clocks and watches, and make calculations based on those observations, conceptually using the model of a needle moving continuously along the real line in the positive direction. But we should not take the model too seriously: if we are told that the new record for running a mile is 3 minutes and 37.65 seconds, we should keep in mind that that is an approximation, as will be any response, no matter how accurate the timepiece. If in the next race, the same runner runs the mile in 3 minutes and 37.65 seconds according to the chronometer available, we say that the time is "the same," but we can neither claim that this is precise, nor that one race was run faster than the other - unless some other observer had a more precise chronometer.

Students should understand the power they have when they are able to move fluently between the verbal description and symbolic representation of a linear context. What does the symbolic representation allow them to do? Symbolic representations allow students to show the relationship between the quantities, solve problems, draw conclusions, make decisions, etc.. Of what must students be careful when they translate to a symbolic representation? Are they clear in what they have defined their variable to represent in the context? What quantities and units are involved? Once solved, can they interpret the solution in the context? Often, when students formulate the algebraic representation and solve it, they struggle to interpret that answer in its context and for confidence that they have even answered the question(s) being asked.

Being able to figure out how to attack problems as they come up, and what protocols to use to solve them, is the central goal of education. Students need to know how the protocols work in order to be able to formulate the problem so that a protocol can be applied. In the following set of problems we concentrate on how to get from a verbal problem to the equation that represents it.

Example 12.

Here is a game, based on the preceding sections, that might attract the students' attention to the uses of algebra. Pick a number from 1 to 20. Add three and double the result. Add 8. Take half of that number. Subtract your original number. You have a 7, right?

Let us analyze this "trick." The subject picks a number; since it is a number but we do not know it, we call it N. Then the subject is asked to perform a set of operations, leading to the following list:

Operation	*Result*
Add 3 :	$N + 3$
Double :	$2(N + 3)$
Add 8 :	$2(N + 3) + 8$
Take half :	$N + 3 + 4$
Subtract original number :	$3 + 4$

So when you, the "magician," reveal that the number in hand is 7, we see that it is algebra, not magic. If this trick is played several times, it is a good idea to change the number 8 to another even number $2K$. Then the end result is "revealed" to be $3 + K$.

A variant of this that may seem a little more magical is this: at the $N + 3 + 4$ stage, ask for the number the subject now has. When this is received, mentally subtract 7 and says: "Your original number was ...."

Example 13.

a. Fred, Jon and Brody pooled their resources for a \$150 room at a resort. Jon put in twice as much as Fred, and Brody put in \$10 less than Fred. How much did each put in?

In this problem we are asked for the amount each put in the pool. In order to pick the "unknown," it may be necessary to read the problem more carefully. The statements all relate the unknowns to Fred's contribution, so we should select that as the primary unknown; let's call it F. Now James put in twice as much as Fred, so he put in twice F; that is, his contribution is $2F$. Brody put in \$10 less than Fred, so his contribution is $F - 10$. The sum of these numbers is \$150, and this becomes the equation:

$$F + 2F + (F - 10) = 150 .$$

We now assign this to our assistant, an expert on the preceding two sections of this chapter, who comes up with the answer : $F = 40$, so Fred put in \$40, Jon put in \$80, and Brody put in \$30.

Here is a slightly more complicated variant:

b. Fred, Jon and Brody pooled their resources for a \$150 room at a resort. Jon put in twice as much as Fred, and Brody put in \$10 less than Jon. How much did each put in?

We are still looking for three numbers, but it is no longer true that the other two are directly related to Fred's contribution. But Brody's is related to Jon's, and Jon's is related to Fred, so it still all comes down to Fred. Again, if we denote Fred's contribution as F, then that of Jon is $2F$. Now Brody put in \$10 less than Jon's, which is $2F$, so Brody's contribution is $2F - 10$. Now add them up to get \$150: $F + 2F + 2F - 10 = 150$, leading to the result $F = 32$.

This problem illustrates several possible sources of confusion, centering around the interpretations of statements like "*A* is four less than *B*" and "*A* is twice *B*." As for the first, it is a problem of the language: there is a significant difference between the statements "*A* less four is *B*," and "*A* is four less than *B*" The students may have trouble with the linguistic difference; this can be resolved by testing with real numbers: take $A = 5$ and $B = 1$. Now, which is true: "5 less 4 is equal to 1" or is "5 is 4 less than 1"? The equation expressing the first statement is "5-4=1," which is true, and for the second we get "5=1-4," which is false. In the same way, we test the second statement: *A* is twice *B*. Take $A = 4$ and $B = 8$, put them in the statement, and pick the version that is true.

Example 14.

A salesman at the XYZ car dealership receives a salary of \$1,000 per month and an additional \$250 for each car sold. How many cars should he sell each month so as to earn \$8,000 in a month?

Solution. The way to the solution is to find out the relationship between income and number of cars sold. Test some numbers: If no cars are sold, the salary is just the base \$1000; if one care is sold, the earnings are \$1250. Look at some more test cases to discover the pattern:

a. if this salesman sells 4 cars, his income for that month is: 1000 + 4(250);

b. if this salesman sells 12 cars, his income for that month is: 1000 + 12(250);

c. if this salesman sells *N* cars, his income for that month is: 1000 + *N*(250).

The last expression simply amounts to recognizing that the computation for 4 or 12 or "no matter how many" cars sold is the same.

In this problem we want the income to be \$8,000, so we look at **c.**, set it equal to 8000 and hand it over to our algebra assistant for the solution.

Example 15.

Lucinda is on the school track team; she can run 8 miles in an hour. Her younger sister Josefa isn't yet a runner, but can walk at a pace of 3 miles an hour. Josefa left their home forty minutes ago heading toward downtown, and Lucinda wants to catch up with her. If she runs how long will it take to catch Josefa? How far will they be from home?

Solution. First, we focus on the first unknown that we have to determine: in this problem it is the time Lucinda needs to catch Josefa; let's call that T, in hours. In that time Josefa has walked $3T$ miles and Lucinda will have run $8T$ miles. So, after T hours, Lucinda is $8T$ miles from home; but since Josefa was already 2 miles down the road when Lucinda started, Josefa is $3T + 2$ miles from home. When they actually meet these two distances have to be the same, giving us the equation

$$8T = 3T + 2$$

The solution of this equation is $T = (2/5)$. Now, it is important to recall what this means: what was T and what are the units for $2/5$? As soon as the algebra takes over, meaning of the symbols becomes irrelevant, but when we've solved the algebraic equation we must return to the meaning of the symbol to fully understand what we have discovered: T is $2/5$ *of an hour*, or 24 min utes. . It takes energy, once the "math" is done, to return to focus to the problem. But when we do, we see there is another part of the problem: How far away have they gone. Well, Lucinda ran for 2/5 of an hour and she runs at 8 miles per hour, so she ran (2/5)8 miles, which is five and a third miles.

Example 16.

My new hybrid car can get 35 miles to the gallon. Gas costs \$3.25 per gallon. San Francisco is 825 miles from here. How much will I spend on gas to drive to San Francisco?

Solution. Let C be the cost of driving there. We have to relate C to miles, denoted by M, and the information we are given is how C relates to gallons, denoted by G, of gas: $C = 3.25G$ and how miles relate to gas: $M = 35G$. This tells us that $G = M/35$, and so we can substitute that in the first equation to get $C = 3.25(M/35)$, or $C = 0.093M$. Since we want to go 825 miles, the cost of gasoline will be $C = 3.25(825) = 0.093(825) = 76.725$, or \$76.73.

The preceding example is, in part, an example of a *literal* problem: one in which several quantities are related and we want to express that relationship by a formula. So, in that example, we want to express the cost of gasoline (C) in terms of miles (M), already knowing the cost of gasoline per gallon and the number of miles per gallon; and we ended up with the relation $C = 0.093M$, or 9.3 cents per mile.

Here is another literal problem:

Example 17.

Let's return to example 14, of the salesman at the XYZ dealership. We saw (see part c)) that if the salesman sells N cars in a month, then his compensation is $1000 + 250N$. The salesman may ask: how many cars do I have to sell to have an income of C in a given month?

Solution. The relationship between compensation (C) and number of cars sold (N) is $C = 1000 + 250N$. The salesman wants to know what N should be to attain a certain value C, so he wants a formula that calculates N, given a value of C. These are the steps:

$C = 1000 + 250N$: this is the starting relationship.

$C - 1000 = 250N$, subtract 1000 from both sides.

$\frac{C - 1000}{250} = N$, divide both sides by 250.

This is the formula to calculate the number of cars to be sold to earn C dollars. So, for example, if the salesman wants to earn \$25,000 in a particular month, he must sell $N = (25000 - 10000)/250 = 96$ cars.

To summarize: here is a procedure for solving problems:

1. Read the problem carefully, making sure to identify the unknown(s).

2. Recognize the information in the problem that can be translated into mathematical expressions or equations.

3. Apply the rules for solving linear equations.

EXAMPLE 18.

The conditions of John's job are that he can work whenever he wants to, but in any day he works, he receives no compensation for the first three hours of work, and $20 per hour for each subsequent hour. On one particular day he wants to buy a special shirt for $190. He has $70 in his pocket. How many hours must he work on that day to be able to buy the shirt?

SOLUTION.

1. What we want to find is the number of hours to work so that the income, together with the $70 he starts with comes to $190. Let N be the unknown: the number of hours he has to work.

2. For N hours of work, he receives no income for the first three hours, and $20 for each subsequent hour. Thus he receives $20 for each of $N - 3$ hours.

3. After N hours of work he has earned $20(N - 3)$. That added to the $70 he started with is to provide the $190 needed to buy the shirt. Thus N must satisfy

$$20(N - 3) + 70 = 190.$$

Our assistant plugs and chugs and finds that $N = 9$. John must work 9 hours in order to have enough money to buy the special shirt.

Chapter 2
Exploring Linear Relations

In the preceding chapter we completed the topic of finding solutions of a linear equation in one unknown. In chapter four we will turn to this study of techniques to find solutions for a pair of linear equations in two unknowns. But now we want to turn to another thread started in previous grades, that of representing and understanding linear relations in two variables. Notice the change in language: from *equation* and *unknown* to *relation* and *variable*. This is a significant change in objective: from that of finding specific numbers that satisfy given conditons, to that of understanding how conditions on the relation of two variables determine how they behave with respect to one another. In seventh grade students studied the properties of a proportional relation between two variables; in this chapter we turn to linear relations between two variables. A significant tool is the graphical representation of a linear relation by a straight line, leading to the correspondence between *rate of change* (for the relation) and *slope* (of the line).

There are two ways to bring together the study of proportional relations and the solution of linear equations in order to understand linear relations, one emphasizing geometric aspects and the other emphasizing the algebra. Algebraically, linear relations are generalizations of proportional representations: we replace the equation $y = mx$ by the equation $y = mx + b$. The commonality between these is that the *rate of change* of y with respect to x is a constant; the difference is that for a proportional relation, the quotient y/x is constant, and is called the *unit rate* of y with respect to x. For a linear relation, the quotient y/x is not constant unless $b = 0$ Here, b is considered the *initial value* of y; that is, the value of y corresponding to $x = 0$. Geometrically, linear relations and proportional relations are both represented by straight lines; the difference is that the graph of a proportional relation goes through the origin, while the graph of a linear relation goes through the point $(0, b)$, called the y-*intercept*. So, a proportional relation is a special case of a linear relation. In particular, if we slide the graph of $y = mx + b$ by the amount b, we get a line through the origin, and thus the graph of a proportional relation. This just realizes the fact that in the linear relation $y = mx + b$, the quantities $y - b$ and x are proportional, with m the constant of proportionality.

The facts that there are these two ways of developing the subject of linear relations, that both approaches are important, and that the differences are subtle, create a learning issue: the student has to assimilate the two approaches at the same time and appreciate the subtle differences between them. Our solution is to present both approaches, the geometric in the workbook and the algebraic in the foundation. This directly exposes the two approaches and gives the teacher the freedom, and obligation, to develop them simultaneously in a way that works best in that classroom.

Here we begin by continuing the study of proportional relations from seventh grade, focusing on the *unit rate* as a rate of change of one quantity with respect to the other. There will be a shift in language as we move from calculating values of quantities in a proportional relation, to the study of the relation itself. For example, we now consider the unit rate as the *constant of proportionality* of the relationship in order to emphasize that it is what remains *constant* while the measure of the quantities vary, and therefore are called *variables*. We observe that the graph of y vs. x, when y and x are in a proportional relation, is a straight line through the origin.

Then, we return to the study of linear expressions, but this time in the form of the function $y = mx + b$ (although we do not introduce the word "function" until chapter 3). We observe that the graph of y vs. x is a straight line that crosses the y-axis at the point $(0, b)$. By examination of tables of values and the graph, we observe that, although the variables are not proportional, their changes from one measurement to another are proportional; that is, the quotient of the change in y values with respect to the x values is constant (independent of the points chosen for the computation). This is called the *rate of change*. The constancy of the rate of change along the graph is a defining property of a straight line, as we shall see in section 3. The move from proportional relations to linear relations, and the accompanying shift from *unit rate* to *rate of change* is subtle and may be difficult for students to appreciate at this time. For this reason, we feel that it is essential to develop the subject in contextual examples, moving the the algebraic formulation in the next chapter.

Although we have observed that the graph of a linear relation is (to be precise, *appears to be*, since we can only plot a finite number of values) a straight line, we still need to understand why this is so. In addition, we need to understand why a straight line is the graph of a linear relation. For this, we seek an algebraic characterization of a straight line, and to get there we have to begin with geometric ideas. Here we introduce *dilations*: transformations of the figures in the plane that retain "shape" but not "size." These properties will be examined in detail in chapter 9; for the present purpose it suffices to observe that a dilation takes a right triangle with horizontal and vertical legs to another such triangle, and that the lengths of the corresponding sides of the triangles are proportional.

If we draw a line in the plane (that is neither horizontal, nor vertical) and then pick two points on the line, the segment of the line between those two points is the hypotenuse of a right triangle with vertical and horizontal legs. This we call *the slope triangle* for that segment. If we now draw the slope triangle for another pair of points, we can exhibit a dilation that takes one triangle to the other. The fundamental property of dilation is this: the length of line segments and the length of the images are proportional, with constant of proportionality the *factor* of the dilation. We conclude from this that the slope of any slope triangle on a given line is constant. In other words, the slope of a segment on the line is constant, and this is called the *slope* of a line. The logic of going from showing that any two computations give the same slope to the statement that slope is constant along the line is a bit subtle, and it might be a good idea to create other examples of that logic.

This leads directly to a way to calculate the *equation of a line*, which is the algebraic expression of the relation between the variables y and x graphically expressed by saying that they lie on a line. The outcome, which will be explored in detail in the next chapter is this: for a line L and two points P and Q on L, construct the slope triangle whose hypotenuse is the segment PQ. Let m be the slope of that segment. Now, let $(0, b)$ be the point on the y-axis that lies on the line. Then the equation of the line is $y = mx + b$.

Figure 1 illustrates the geometry in this discussion. We show the cases for both positive and negative slope, to emphasize that slope is not the ratio of the lengths of the sides of the slope triangle, but the ratio of the *changes* in the variables. Thus, when the change in y is negative for the corresponding increase in x, then the slope will be negative.

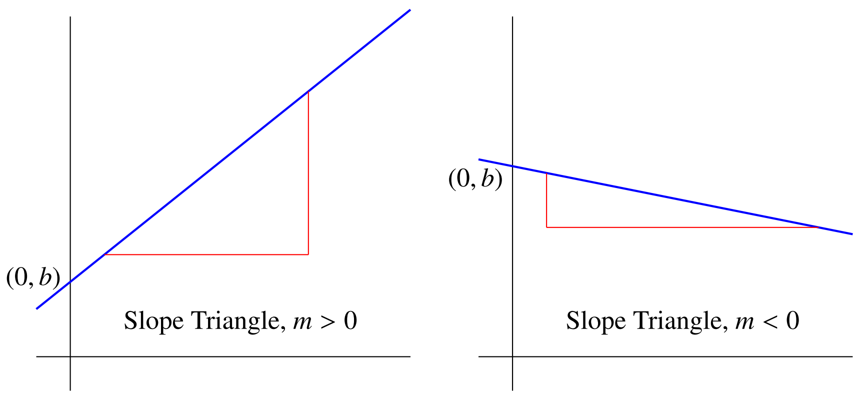


Figure 1

2.1 Linear Patterns and Contexts

Proportional Relationships

Graph proportional relations, interpreting unit rate as the slope of the graph, which is a straight line. 8.EE.5.1

Compare two proportional relationships represented in different ways (tables, graphs, equations). 8.EE.5.2

The ideas of ratio and proportion were introduced in grade 6 and further developed in grade 7. In this section, after a brief review of the development of these ideas, we move on to the relation "proportional" (from the focus on the values of variables "in proportion." This complements the gradual move away from the language of "unknowns" and "equations" to that of "variables" and "relations."

In grade 6 the concept of *ratio* is introduced as a way of describing a relation between two collections of objects without reference to the actual size of those collections. So we may say that, in U.S. population, the ratio of minors to adults is 2:5, meaning that there are 5 adults for every two minors. This knowledge tells us, for example, that if we collect together a random group of people of size 140, we should expect 40 of them to be minors.

An old joke says that a shepherd keeps track of the herd by counting the legs and then dividing by 4. In terms of ratios, this expresses the fact that the ratio of sheep to sheep legs is 1:4. Actually, in Utah and Nevada where there are sheep herds numbering in the thousands, the shepherd keeps track of the herd by counting the black sheep and then multiplying by 40. This works for two reasons. First, sheep are social animals congregating in groups, so a count of sheep that estimates the actual number of the sheep within the size of a group has not missed any groups of sheep (maybe a stray lamb or two, but then the mother ewe will notify the shepherd of her distress). Second is that the ratio of sheep to black sheep is, for reasons of genetics, 40:1.

The concept of ratio (used mostly in counting individuals in particular sets) leads to the concept of *proportion*, which is more convenient than ratio for quantities in question can take on all numerical values, not just integral values.

- Given two quantities x and y, they are said to be *proportional* if, whenever we multiply one by a factor r, the other is multiplied by the same factor, r. For example, if we double the variable x , then y also doubles.

If two quantities are measuring the same physical attribute, they are going to be proportional. When we measure the length of a rod, we may do so in yards (Y), or in feet (F). Since these are measures of the same physical characteristic, they have to be proportional: if the rod triples in size, then its measure in feet or in yards also triples in size. A yard is defined as being 3 feet long, so we say that the ratio of yards to feet is 1:3. This can be rephrased as a proportional relationship with the *unit rate*: 3 feet per yard.

- If quantities y and x are in proportion then the *unit rate* of y with respect to x is the amount of y that corresponds to one unit of x. If m is the unit rate, then for any value of x, the corresponding y value is mx. If we interchange the roles of y and x, we would speak of the unit rate of y with respect to x . These two numbers are inverses of each other.

Since the unit rate of feet to yards is 3, the unit rate of yards to feet is 1/3.

Example 1.

There are 5280 feet in a mile. How many yards are in a mile?

Solution. $1 \text{ mile} = 5280 \text{ feet} \times \frac{1 \text{ yard}}{3 \text{ feet}} = \frac{5280}{3} \text{yards} = 1760 \text{ yards}.$

In seventh grade, the unit rate is reinterpreted as the *constant of proportionality*. This corresponds to the change of focus from specific instances of a proportional relationship to that of the relationship itself. This leads to the equation $y = mx$, where y and x are the quantities in the proportional relation, and m is the constancy of proportionality. When two variables are proportional, all we need to know is one specific pair of values (x_0, y_0) in the relation to be able to compute all such pairs of values, for the ratio y_0/x_0 gives us the value of m. Graphically this is clear: if we know a pair of values (x_0, y_0) in the relation, all pairs (x, y) in the relation lie on the line joining $(0, 0)$ to (x_0, y_0). So all we need to do is to draw the line joining the origin to the given point, (x_0, y_0).

Example 2.

The concepts of *ratio*, *constant of proportionality* and *unit rate* seem interchangeable, since they can all be represented by the same fraction, and this causes a lot of confusion with students. The way to address this confusion is to first understand that they are interchangeable, and are used in different ways in different contexts. So the second step in addressing this issue is to understand that the fundamental difference among these concepts is that they present different ways of looking at a problem in context, and that one has to learn how to decide which interpretation is relevant for a given context. Let us illustrate.

In basketball, it is necessary to have 12 players in a roster. In a particular district in Eastern Utah, the middle school basketball league has teams that are made up of boys and girls. For fairness, it is decided that each team must have 7 girls and 5 boys. This tells us that the ratio of girls to boys in the basketball league is 7:5. The relation, girls to boys in the basketball league is a proportional relationship, with constant of proportionality "girls to boys" equal to 7/5.

Question 1. The district decides to have 8 teams in the league. How many girls and boys are there in the competition? This problem guides us to think in terms of the ratio 7:5: since there are 8 teams, each of which has 7 girls and 5 boys, the total number of players are $8 \times 7 = 56$ girls, and $8 \times 5 = 35$ boys.

Question 2. There are 45 boys eligible for basketball. How many girls are needed to to complete the league? Here we want to think in terms of the constant of proportionality, which is 7/5. So the number of girls needed is 7/5 of the number of boys available; that is, $(7/5) \times 45 = 7 \times 9 = 63$.

Here is a different problem: my grandfather drives at exactly 30 miles per hour.

Question 1. If Gramps drives 5 hours, how far does he go? Here, we think of unit rate: the rate of miles per hour is 30. Since

$$\text{miles} = \frac{\text{miles}}{\text{hours}} \times \text{hours}\ ,$$

he traveled $30 \times 5 = 150$ miles. Question 2: Gramps wants to drive to St. George from Logan; that is 440 miles. How long will it take him at that rate. Here we want to convert to minutes, and the concept of ratio: the ration of minutes to miles is 2:1. So to drive 440 miles, takes Gramps 880 minutes, or 880/60 = 14. 6667, or 14 hours and 40 minutes.

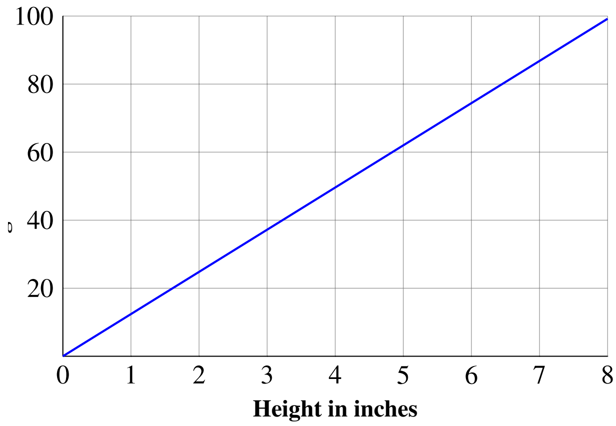
Example 3.

To illustrate the development of a proportional relationship, consider measuring the amount of water in a cylindrical container (a glass or can). If we put a quantity of of water in the cylinder, we record the height of the column of water, H, and the weight W of the column of water. Both of these are measures of the amount of water: if we double the amount of water, both the height and the weight double.

Suppose that this experiment is done with a quantity of water, filling the cylinder a bit at a time, and each time, measuring both H and W. A table of the data would look something like this:

Height	0	2	3	4	6	8	inches
Weight of Container	2.5	2.5	2.5	2.5	2.5	2.5	ounces
Measured Weight	2.5	27.3	39.7	52.1	76.9	101.7	ounces
Weight of Water	0	24.8	37.2	49.6	74.4	99.2	ounces

Notice that, we have accounted for the weight of the container itself by first measuring it empty, and then subtracting that weight from the weight of the container and water at each measurement. We now graph the these data, plotting height along the horizontal axis and weight on the vertical: The graph appears to be a straight line, giving confirmation of our hypothesis that the height of the column of water and its weight are proportional. We can calculate the unit rate of change using any one of the measurements: for example, 4 inches of the water weighs 49.6 ounces, so we have 12.4 ounces per inch of water. This is expressed by the relation $W = 12.4H$. In an actual experiment there always will be slight variations due errors or estimation, given the accuracy of the instruments used. So, the rates computed from each measurement may differ slightly. We will return to this in the statistics chapter.



Graph of Example 2 Data

In summary, the height of the column of water in the container and its weight are two different ways of measuring the volume of the column of water. Since the volume of the water is the same no matter how it is measured, the measurements are related. Similarly, yards, feet, inches and meters are different ways of measuring lengths; ounces, pounds, grams are different ways of measuring weight. All these relations have the property that a doubling or halving of the object (volume of water or length of stick) has the effect of doubling or halving the measure. In fact if the amount of the object is changed by the factor a, then any measure of the object also changes by the factor a. When quantities are related in this way, we say that they are *proportional*.

Example 4.

If we are told that x and y are in the relationship $y = 7x$, then (1,7) , (2.5, 16.5), (8,56) are all in this relationship, because the ratio of the y value to the x value is always 7.

Example 5.

There are 5280 feet in a mile, so Feet/Miles = 5280, or Feet = 5280×Miles. To find out how many feet are in a quarter mile, let f represent that number of feet. Then we have $f = 5280(1/4) = 1320$ feet. In yards, that is 1320/3 = 440 yards.

Example 6.

We have made measurements of two quantities, and formed this table:

x	0	2	4	5	7	8	10
y	0	3.6	7.2	9	12.6	14.4	18

The graph of these data appears to be a straight line through the origin suggesting a proportional relationship: Notice that whenever the value of x doubles, so does the value of y, and that a change in x of 1 unit is accompanied by a change in y of 1.8 units. Finally, when we calculate the quotient y/x for any pair of points, we always get the value 1.8. This can be phrased this way: the proportional relationship $y = 1.8x$ models the given data.

- If quantities y and x are in proportion then the graph of pairs (x, y) in this relation will be a straight line through the origin. That line is characterized by the assertion that y/x is constant, and in fact, is the constant of proportionality. In terms of the graph, we call this its *slope*.

Linear relationships

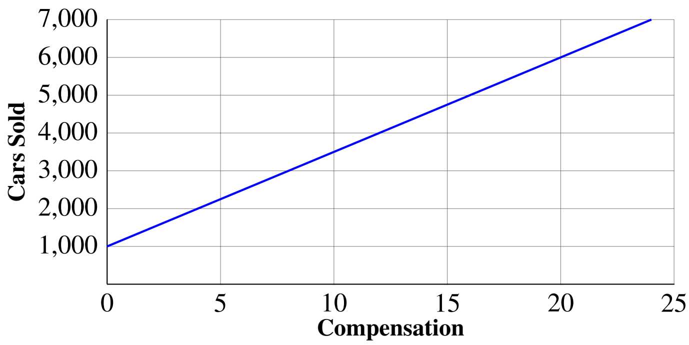
Construct a function to model a linear relationship between two quantities. 8.F.4

In chapter 1 we concentrated on solving linear equations of the form y = *linear expression*, where y is either a number or another linear expression. We also compared two linear expressions by graphing them (see the figure on page 6 of chapter 1)), and found that the graph of each linear expression is a line. In this chapter our goal is to see why there is this correspondence between linear expressions and lines. Let's start by taking another look at example 13 of Chapter 1.

Example 7.

A salesman at the XYZ car dealership receives a base salary of \$1000/month and an additional \$250 for each car sold. How many cars should he sell each month so as to earn a specified amount each month? There we ended up with this formula: $C = 1000 + 250N$, where N is the number of cars sold in a month, and C is the compensation received. Let's make a table for some possible values of N and then graph the result:

N	0	4	8	12	16	20
C	1000	2000	3000	4000	5000	6000



Cars Sold and Compensation

The graph (see the figure above) is a straight line that does not go through the origin: even if the salesman sells no cars, he receives the base salary of \$1000. Also note that to each increment of 4 cars sold, the salesman receives an increase of \$1000. In particular we can say that *the increase in income is to the increase of number of sales as 1000:4* giving us a unit rate of \$250 in compensation per unit of cars sold. This is just the coefficient of N in the equation $C = 1000 + 250N$. To restate this: the number 250 expresses a relation between the variables N and C, even though the variables are not proportional. It is the *change* in C that is proportional to the *change* in N at the ratio 250:1.

Let's look at a few more examples to emphasize this point and to see how students should be able to extend this idea.

Example 8.

At the statewide championship game, each player on each team receives five complimentary tickets, and can buy additional tickets at \$20 each. Carlos wants 8 tickets and Louis wants 16 tickets. How much does each pay for the full set of tickets?

Solution. One might to say that, since Louis is getting twice as many tickets, he has to pay twice as much. But that would be a mistake, the cost is not proportional to the number of tickets, but cost *is* proportional to the number of tickets *in excess of 5* . In this situation, they each get 5 complimentary tickets, so Carlos pays for 3 tickets and Louis pays for 11 tickets. At \$20 apiece, Carlos pays \$60 and Louis pays \$220.

By applying this thinking to the general case, we can write down a formula for the cost C of N tickets for any player. If a player wants N tickets, he gets 5 free and pays \$20 each for the remaining tickets. There are $N - 5$ remaining, so the cost is $C = 20(N - 5)$ or $C = 20N - 100$. The form of these equations tell different things, both interesting. The first ($C = 20(N - 5)$) tells us that the cost is proportional to the excess of tickets above the first 5. The second tells us that the cost is \$20 per ticket, less \$100 for the free 5 tickets. Note that these equations make sense only for $N \geq 5$; players don't get refunded if they have less than 5 friends. In figure 2 we have graphed this relationship:

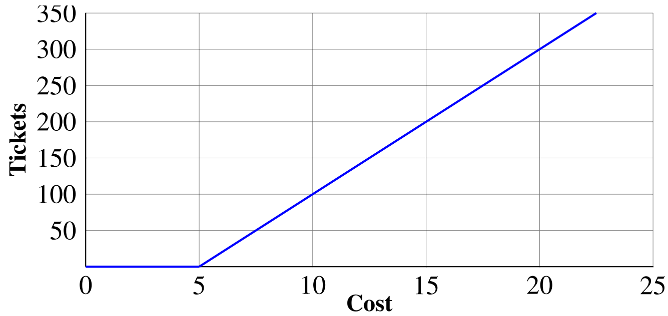


Figure 2

For any table where rate of change of one variable with respect to the other is **not** constant, the graph of these data will **not** be a straight line. Students will begin to explore these types of relationships more in Secondary 1. In 8th grade students simply need to recognize if a relationship is linear or not.

Example 9.

I have a 120 gallon steel drum full of water to keep my garden thriving through a long dry spell. Each day I use four and a half gallons watering my plants. How much water do I have in the drum after 10 consecutive dry days? After d consecutive dry days? How long can I last without rain or refilling my drum?

Solution. If I use 4.5 gallons of water each day, in 10 days, I use $4.5(10) = 45$ gallons of water, so there are 75 gallons of water still in the drum. After d days there are still $120 - 4.5d$ gallons in the drum. Using the symbol w to indicate the amount of water in the drum, this gives me the relation

$w = 120 - 4.5d$. Figure 3 is the graph of that relation.

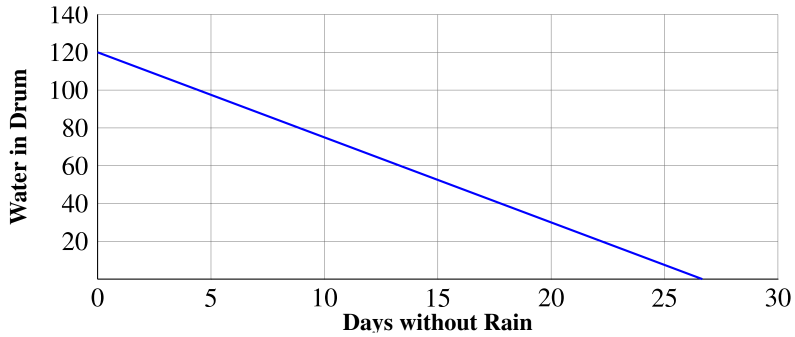


Figure 3

Notice that this time the line is pointing downward, that is because the amount of water in the drum decreases as the number of days increases. If we calculate (change in w)/(change in d) for any two points on the graph the result will be -4.5, indicating that each day we have 4.5 gallons less in the drum. It is important to note that language plays a role here: The word less accounts for the negative sign: it would be wrong to say that "each day we have -4.5 gallons less in the drum." What is correct is "the ratio of change in water to the change in day is -4.5 gallons to 1 day."

Example 10.

Consider the image in Figure 4:

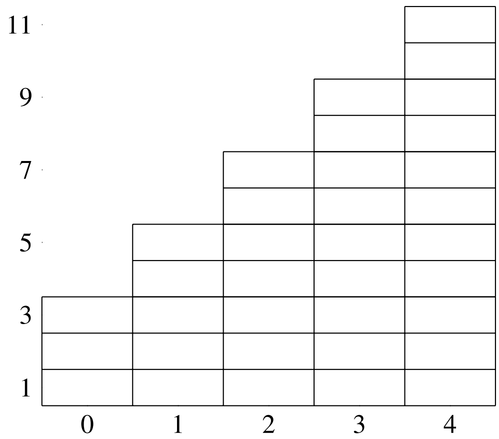


Figure 4

There is a pattern here: each time we move to the right by one unit, the height of the stack increases by 2. We have labeled the axes in figure 4 with x representing the number of moves to the right, and y the height of the stack. So, the first stack has the value 0, indicating that there are no moves to the right yet, and the last stack is 4 moves to the right. The height of the stack starts at 3, and with each move to the right, increases by 2. This tells us that the algebraic relationship is $y = 3 + 2x$.

Example 11.

The Timpanooke trail (see the image) is a 7 mile trail from the foot of Mt. Timpanogos (at 7200 feet) to the peak (at 11900 feet). The trail has three different segments: the first is a three and a half mile horse trail with a steady altitude gain; the second is a two and a half mile traverse across a nearly level basin, and the last is a one mile steep climb to the peak. The accompanying table shows the altitudes at each of these transition points, and the time it takes an average hiker to cover each leg. Make two graphs; on both the horizontal axis is "miles" and on one, put "altitude" on the vertical axis, and on the other, "hours." Calculate, for each leg of the trek, the rate of change of altitude with respect to miles, and of hours with respect to miles. Compare and contrast the two graphical representations. Can you explain the similarities in the two graphs? This activity, of looking for similarities and differences among graphs will be studied in depth in Secondary 1.

http://farm3.static.flickr.com/2659/3692964206 215e54c7d7.jpg

Timpanooke Trail				
Altitude	7200	8700	10,700	11,900
Miles	0	3.5	6	7
Hours	0	2	3.5	5

In the above examples of linear relations, we have seen from the tables of values that the rate of change in y with respect to x is constant. That constant is positive if the graph points upwards as we move from left to right, and negative if the graph points downward. If the graph is horizontal, there is no change in y, so the rate of change is 0. It seems to always turn out that the graph of a linear relation is a straight line, but this is something we cannot yet explain. It is important to always keep in mind that the subject of mathematics - indeed every science - is not to just record observations, but to study them enough to be able to explain them. That is the only way that the working scientist can make progress, be it in mathematics, medicine or space travel. In classroom discussions, it is important, where relevant, to be precise about *observation* vs *understanding*. Here we're observing that the relationship is linear, we have not proven this fact.

So, what is the property of a line that guarantees that it will be described by a relation of the form $y = mx + b$? What we know about a line is that it is determined by two points: place a straight edge against the two points and draw the line. A line is also determined by a point and a direction: lay the straightedge against the point, and set it in the intended direction and now draw the line. To relate these to a condition on linear relations, we need to find an algebraic way of expressing this geometric, constructive criterion. This is done with the concept of *slope* of a line and its relation to rate of change.

Section 2.2. Slope of a Line

Describe the effect of dilations ... on two dimensional figures using coordinates. 8.G.3: that the image of a line is a line parallel to it; that, under a dilation a line segment goes to a line segment whose length is the length of the original segment multiplied by the factor.

Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation $y = mx$ for a line through the origin and the equation $y = mx + b$ for *a line intercepting the vertical axis at b. 8.EE.6.*

In order to respond to this last standard in the way it is stated, a chapter on transformational geometry up to similarity would have to precede this chapter. We felt that it is important in eighth grade to begin the year by completing the set of ideas around linearity, and that an initial chapter on geometry would be a diversion from this main point of 8th grade mathematics. Since all that is needed to understand the main fact about slope are the two properties of dilations cited above, we decided to minimize the geometry to these facts, and then return to the relation of the rate of change of a linear function and the slope of the graph of that function: they are the same. Dilations are connected to scaling, so it could be useful to recall at this time that discussion in 7th grade.

A *dilation* is given by a point C, the center of the dilation, and a positive number r, the factor of the dilation. The dilation with center C and factor r moves each point P to a point P' on the ray CP so that the ratio of the length of image to the length of original is r: $|CP'|/|CP| = r$.

Figure 5 illustrates a dilation. In the figure, the center of the dilation is C, and its factor is r. We have exhibited 3 original points, P, Q, R and their images under the dilation P', Q', R. The letters a, b, c are the distances of P, Q, R from C.

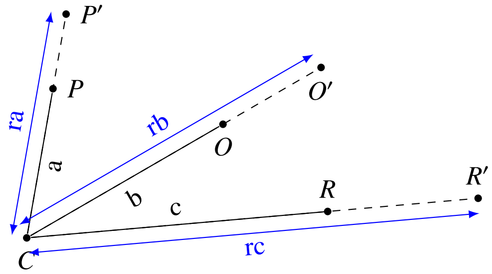


Figure 5

Example 12.

In figure 6 we illustrate the effect of a dilation with center $C = (0, 0)$, and factor $r = 2.5$ on a triangle in the first quadrant of a coordinate plane.

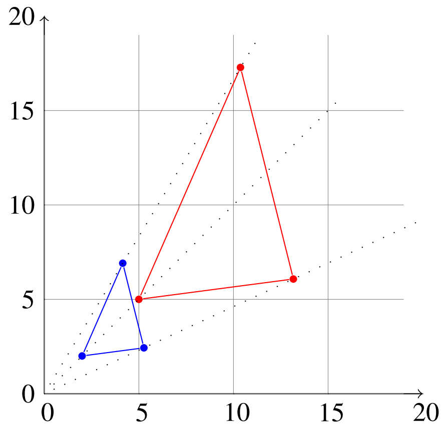


Figure 6

Observe the connection of this image with those in the 7th grade discussion of scale drawings. Note also that a point $(x, 0)$ is moved to the point $(2.5x, 0)$, and a point $(0, y)$ is moved to $(0, 2.5y)$. In fact, any point (x, y) is moved out to the point on the lines through the origin and that point whose distance from the origin is 2.5 times that of (x, y). That the coordinates of this point is $(2.5x, 2.5y)$ is easily observed, and gives the coordinate description of a dilation with center the origin. However, the understanding of why this is true needs the Pythagorean theorem, to which we will return in chapter 10. For now, students should work many examples of this type to conclude that

- In a coordinate plane, the dilation with center the origin and factor r is given by the coordinate rule $(x, y) \rightarrow (rx, ry)$.

Being able to express a transformation of the plane in terms of coordinates provides an algebraic tool to help work with conceptual understanding of dilations, but it is not as important at this stage as being able to understand the properties of dilations. Students should play with the concept sufficiently to accept these properties as intellectually plausible:

Properties of the dilation with center C and factor r:

a. If P is moved to P', then $|CP'|/|CP| = r$. That is, the distance of P' from C is r times the distance of P from C

b. If P is moved to P' and Q is moved to Q', then $|Q'P'|/|QP| = r$. That is, under a dilation, the length of any line segment is multiplied by the factor of the dilation.

c. The dilation takes parallel lines to parallel lines.

d. A line and its image are parallel.

The first is part of the definition of a dilation. The second, that *every* length, not just those on lines through the center, is multiplied by the factor of the dilation, can be confirmed in examples - enough so that students accept this conclusion. The last two about parallelism are central properties of dilations. They are easy to observe through examples, and they are intuitively plausible. They follow from the fact that parallel lines do not intersect. As for **c**: if two lines do not intersect before the dilation, their images cannot intersect; this would imply that the dilation takes two different points to the same point; this is not possible. And for **d**, if a line and its image intersect, that point of intersection was not moved by the dilation. Unless $r = 1$, the only point not moved by the dilation is the center. In chapter 9 we will return to this subject and see that property **b** is a consequence of the other three properties in this statement.

In the preceding section we observed that the graph of a proportional relation is a straight line through the origin, we now turn to understanding why this statement and its converse is true. The key here is the above set of properties of dilations. Let's start with the statements that we want to understand:

- A non-vertical straight line through the origin is the graph of a proportional relation $y = mx$.
- The graph of the proportional relation $y = mx$ is a non-vertical straight line through the origin.

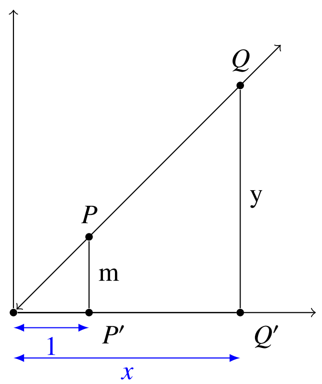


figure 7

We start with the first statement, and then show that it implies the second. In figure 7 we have drawn a typical line L through the origin. P is the point whose first coordinate is 1 and whose second coordinate is m. Q is any other point on the line with coordinates (x, y). We introduce the dilation with center the origin that takes the point P' to Q'. Since the length 1 goes to the length x, the factor of the dilation is x. Now, the dilation takes the vertical line PP' to a parallel, and therefore also vertical line through Q'. That line has to intersect L at Q, since the line L is not changed in the dilation. Now, the dilation multiplies the length of PP' by x, so the length of QQ' is mx. But that is y, so we can conclude that $y = mx$. Since Q was any other point on L, we have shown that L is the graph of the proportional relationship $y = mx$.

As for the converse, we start with a proportional relationship $y = mx$. Draw the line through the origin and the point $(1, m)$. By the argument above, this line is the graph of the proportional relationship $y = mx$.

Now, we shall mimic this construction for the general non-vertical straight line. This less to we have similar statements, with “proportional” replaced by “linear,” and “unit rate” replaced by “rate of change.” The statements we want to discuss are:

- A non-vertical straight line is the graph of a linear relation $y = mx + b$.
- The graph of the linear relation $y = mx + b$ is a non-vertical straight line through the point $(0, b)$ (called the y-intercept).

To see why these are true, start with a linear relation $y = mx + b$. Of course, when b = 0, the graph goes through the origin and is a proportional relationship, so, by the above argument the graph is a straight line. Now we could argue as follows: Given the equation $y = mx + b$, first look at the graph of the proportional relation $y = mx$. We now know that that is a straight line L. If we shift that graph in the vertical direction a distance of b units, we still have a straight line L'. We also know that if (x, y) are the coordinates of a point on L', then $(x, y - b)$ are the coordinates of a point on L. So we must have $y - b = mx$; that is: this equation is satisfied by the coordinates of any point on L'. But this is the same as $y = mx + b$, so L' has to be the graph of the linear relation $y = mx + b$.

For the converse, start with a non-vertical line L. Since it is non-vertical it intersects the y-axis in a point $(0, b)$. If we shift this point to the origin, we get a new line L' through the origin which is, therefore, the graph of a proportional relationship $y = mx$. But if (x, y) is on L, $(x, y - b)$ is on L' and so we again have $y - b = mx$ as a relation defining the line L, or what is the same $y = mx + b$.

This argument gives a geometric meaning to the number b: it is the y-coordinate of the point of intersection of the line with the y-axis (the y-intercept). But what is the geometric meaning of the number m?

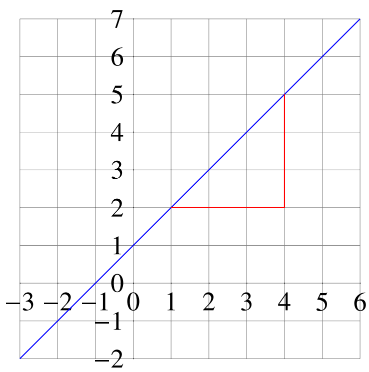


Figure 8

Let us start again with a non-vertical straight line in the coordinate plane, that doesn’t go through the origin, but through some point $(0, b)$ on the y-axis. We know that two points on a line determine the line: just put a straight edge against both points, and draw the pencil along the straightedge. We now want to see how to describe this in terms of coordinates: how do the coordinates of two points on a line determine the relation between the coordinates of any point on the line? This is where slope comes in.

Given two points in the coordinate plane, P and Q, we define the rise to be the difference of the y values from P to Q, and the run to be the difference in the x values from P to Q. The slope of the line segment is the quotient of these two differences:

$$slope = \frac{rise}{run}$$

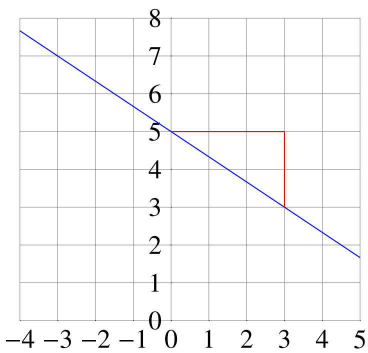


Figure 9

If P has the coordinates (x_0, y_0) and Q has the coordinates (x_1, y_1) this is

$$slope = \frac{y_1 - y_0}{x_1 - x_0}$$

Geometrically, if we draw the triangle with hypotenuse the line segment from P to Q and legs horizontal and vertical - this is the *slope triangle* - the slope is the signed quotient of the length of the vertical leg by the length of the horizontal leg. By *signed*, we mean that the slope is positive if the line points upward as we go to the right, and negative if the line points downward. (see figures 8 and 9).

Note that in the slope computation the differences have to be taken in the same order: if we subtract the y value of P from that of Q, we must subtract the x value of P from that of Q. However, if we interchange the points P and Q, we get the same number. For a vertical line, the denominator in the quotient is zero, so the slope is not defined.

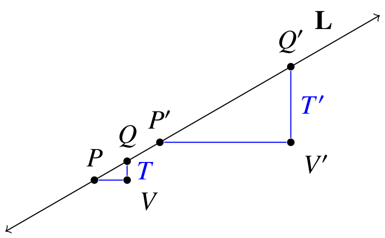


Figure 10

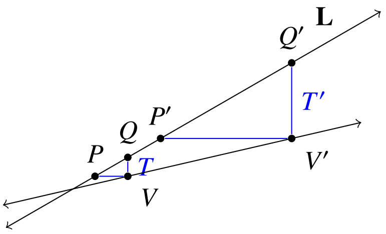


Figure 11

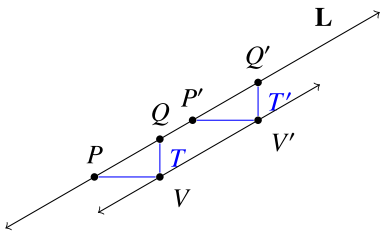


Figure 12

For a horizontal line, the numerator is zero, so, the slope is zero. Since the equation of a horizontal line is of the form $y = b$, this corresponds to the fact that y does not change as we move along the line. What we want to show is this: for a line L, this slope calculation is the same for any two points P and Q on L and is called the *slope of the line*.

Let L be a non-vertical line, P ,Q and P', Q' two different pairs of points on the line, and T and T' the right triangles whose hypotenuses are the given line segments, and whose legs are horizontal and vertical. Label the vertices at the right angles as V and V' (see figure 10). These two triangles appear to be related by a dilation; we want to show that they are. First, if there is a dilation that takes T to T', it must be the case that P goes to P', so the line L is a line through the center of the dilation. Also, V goes to V', so, by the same reasoning V and V' also lie on a line through the center of the dilation. Let L' be the line through V and V'. The point of intersection C of L and L' has to be the center of the dilation (figure 11), and its factor r has to be the ratio of the length of CP' to that of CP. Let's verify that this dilation does take T to T'. First of all, it takes P to P', since that is how r was chosen. Since the dilation preserves "horizontal," and preserves the line L', it takes the segment PV to $P'V'$, and so the ratio of those lengths is also r. Since the dilation preserves "vertical," and preserves the line L, it takes the segment QV to $Q'V'$, so the ratio of those lengths is also r. Thus, in moving from T to T', the length of every side is multiplied by the same factor r, so when we calculate the $rise/run$, the r's cancel, and the quotient is the same for both triangles.

There is one case not covered: the lines L and L' may not intersect; that is, the are parallel. In this case, (see Figure 12) under the shift of P to P', the triangle T slides along these tracks to T' without changing the lengths of the sides.

Section 2.3. The Equation $y = mx + b$.

To wrap up this chapter, we bring together all the preceding material, not simply to summarize it, but also to lead in to the next chapter, a study of linear functions and lines, the purpose of which is to develop flexibility in moving among the representations of linear relations.

- For a line L, for any two points P, Q on the line, the quotient

$$\frac{rise}{run} = \frac{\text{change in } y \text{ from } P \text{ to } Q}{\text{change in } x \text{ from } P \text{ to } Q}$$

  is constant, and that constant is the *slope of the line*.

Example 13.

(0, 5), (2, 9), (−1, 3) are three points on a line. Calculate rise/run for each pair of points.

Solution. First we should verify that indeed the three points lie on a line, using the slope calculation In each case that calculation produces 2 as the slope of the line, for example, taking the third and first points, we have:

$$\frac{3-5}{-1-0} = \frac{-2}{-1} = 2$$

Let us see what this example tells us. Start with two pairs of points; suppose that we find that the slope calculation produces the same number. This does not mean that the two pairs of points are on the same line (what does it mean?). However if there is a point in common to the two pairs, then all three do lie on the same line. This tells us something important. Given a line, pick two points P, Q on the line. Calculate the slope m using these two points. Now take any point X on the plane, and calculate the slope of the slope triangle using the points X and P (or Q). If the result is m, then, by this observation, X is on the line; if it is not m, then X is on the line. We have generated a protocol for deciding whether or not a point X is on the line through P and Q.

Going back to example 12, the line through any pair of these points has slope 2. So, for any point (x, y), if any of the calculations

$$\frac{y-5}{x-0} \qquad \frac{y-9}{x-3}, \qquad \frac{y-3}{x-(-1)}$$

gives the value 2, then they all do, and (x, y) is a point on the line. If any of these computations do not give 2, then none do, and (x, y) is not on the line. So, we have this test for a point (x, y) to be on the line:

$$\frac{y-5}{x-0} = 2$$

By multiplying both sides by x we get $-5 = 2x$, or $y = 2x + 5$. This is called the equation of the line. Instead of choosing the first point, we could have chosen one of the other two getting the test:

$$\frac{y-9}{x-2} = 2, \qquad \frac{y-3}{x-(-1)} = 2$$

No matter what point we choose for the test, after simplification we will always get the equation $y = 2x + 5$.

Chapter 3 starts with an examination of techniques to find the *equation of a line*, beginning with this example.

Chapter 3
Representations of a Linear Relation

The purpose of this chapter is to develop fluency in the ways of representing a linear relation, and in extracting information from these representations. In the first section we shall study linear relations among quantities in detail in each of the realizations: formulas, tables, graphs and context, and develop fluidity in moving among them. Although the Core Standards refers to the concept of *function*, in this chapter we continue this as a discussion of linear relations, which, when of the form $y = mx + b$ is called a function. The transition in thinking from "equations" to "relations" to "functions" is - as was that from "unknowns" to "variables" - subtle but significant. The outcome for the student is to develop a way of seeing functions dynamically, and as expressions of the behavior of two variables relative to each other. For these reasons, we move ahead slowly; in this chapter developing technique in studying linear relations, in chapter 4 concentrating on the simultaneous solution of two equations, and then returning to an in-depth study of functions in chapter 5.

In the second section we go more deeply into the relationship of the geometry and algebra of lines; giving the slope conditions for two lines to be parallel or perpendicular. There are two advantages in introducing this topic now. First, it provides an opportunity to introduce translations and rotations and use their basic properties, and second it gives an application of the idea of slope in comparing two lines.

3.1 Linear relations: creating graphs, tables, equations of lines

Interpret the equation $y = mx + b$ as defining a linear function whose graph is a straight line. 8.F.3.

Determine the rate of change and initial value of the function from a description of a relationship, or from two (x, y) values, including reading from a table or a graph. 8.F.4.

In example 13 of Chapter 2, we considered three points: $(0, 5)$, $(2, 9)$ and $(-1, 3)$, and calculated the rise/run for each pair of points always arriving at the answer: *slope* $= 2$. So, $(2, 9)$ and $(-1, 3)$ are on a line through $(0, 5)$ of slope 2. But there is only one line through $(0, 5)$ of slope 2, so it must be that all three points lie on the same line.

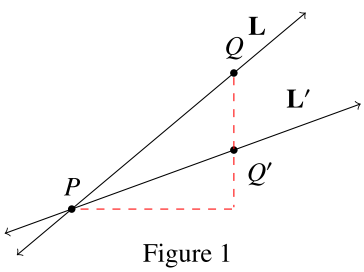


Figure 1

It is intuitively clear that there is only one line through a given point and of given slope, and figure 1 shows us why. The lines L and L' intersect at the point P; we have drawn the slope triangle with a run of one unit for both lines. If the lines L and L' are different, then the rise for these lines is different (otherwise Q and Q' would be the same point), and so the slope is different.

We can also give an algebraic argument. Start with two lines L and L' and let the equation for L be $y = mx + b$, and that for L' be $y = m'x + b'$. If the lines intersect that tells us that there is a value for x at which the expressions $mx+b$ and $mx' + b'$ have the same value (namely the y coordinate of the intersection point). Let x_0 be that value of x, so that $mx_0 + b = m'x_0 + b'$. Now, if the lines have the same slope, $m = m'$, and now subtracting mx_0 from both sides, we get $b = b'$; that is, the lines L and L' have the same equation, and therefore are the same line.

Now, let's pick up where we left off in Chapter 2, and recall the definition of *the equation of a line*. We start with a line L, and a point P on L with coordinates (a, b). If (x, y) is any other point on the line we must have

$$\frac{y-b}{x-a} = \text{slope of } L.$$

Because there is only one line through P of that slope, we also know that if (x, y) is *not* on the line L, this computation cannot give the slope of L. Thus the above equation is a test for a point to be on the line L, and thus is called *the equation of the line* Returning to the points (0,5), (2,9) and $(-1, 3)$, we can use any two to calculate the slope, 2, and then use any of the three points to test for membership in the line:

$$\frac{y-5}{x-0} = 2 \qquad \frac{y-9}{x-2} = 2 \qquad \frac{y-3}{x-(-1)} = 2$$

Thus, for example the point (3,11) is on the line because it passes the test: the slope calculation with any of the given points always gives 2. On the other hand, (6.2) is not on the line, for each of the computations gives a number different from 2. Of course, we don't have to test the slope equation with every point on the line, but just with one point (and maybe another to check the calculation).

The equations above are not in simplest form, and if we clear of fractions and simplify to the form $y = mx + b$ we do get the same values of m and b. This could not be otherwise, for we can identify m and b as characteristics of the line: m is its slope and $(0, b)$ is the intersection of the line with the y-axis. So, when put in simplest form ($y = mx + b$), there is only one equation of the line.

Let's follow this through for each of the above equations, first clearing of fractions, and then isolating y on the left hand side of the equation

$$\frac{y-5}{x-0} = 2 \qquad\qquad \frac{y-9}{x-2} = 2 \qquad\qquad \frac{y-3}{x-(-1)} = 2$$

$$y - 5 = 2x \qquad\qquad y - 9 = 2(x-2) \qquad\qquad y - 3 = 2(x+1)$$

$$y = 2x + 5 \qquad\qquad y = 2x - 4 + 9 \qquad\qquad y = 2x + 2 + 3$$

$$y = 2x + 5 \qquad\qquad y = 2x + 5$$

So, we ask: is (3,10) on the line? We calculate the slope of the line segment between (3,10) and (2,9), and get 1. Thus (3,10) is not on the line. But (3,11) is a point on the line, since $(11 - 9)/(3 - 2) = 2$. More importantly, note that every slope calculation (as those just executed) always simplifies to a unique equation $y = mx + b$.

Once we know the slope of the line, we can use any point on the line to calculate the equation of the line. And if we know two points on the line, we can use those points to compute the slope. Then, using one of the points and the slope, calculate the equation of the line. Restating this: if we know a point on a line and the slope of the line, we can calculate the equation of the line. This corresponds to the geometric fact that a point and a direction determine a line. Next, if we know two points on a line, we can calculate the equation of the line; corresponding to the the geometric fact that two points determine a line.

To sum up: The equation of the line (the test for a point (x, y) to be on the line) can always be written in the form $y = mx + b$, called the *slope-intercept form of the equation of a line* because m is the slope, and $(0, b)$, the y intercept is on the line. No matter what points on the line we choose for the calculations, the point-slope form of the equation will always be the same.

One last important point, to which we will return in the next section and again in Chapter 5. An equation of the form $y = mx + b$ describes a process: As the value of x changes, the value of y changes along with it. And, the slope is calculated as the quotient of the change in y by the change in x between any two points on the line:

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

so it is the *rate of change* of y with respect to x, and the fact that the graph is a line tells us that the rate of change is constant. Since b is the value of y when $x = 0$, we also refer to b as the *initial value*.

Example 1.

Masatake runs at a constant rate. At a recent marathon, his friend Jaime positions himself at the 5 mile marker, and Toby is at the 8 mile marker. Masatake passes Jaime at 8:40 AM, and then passes Toby at 9:01 AM. If Masatake can keep up that rate, at what time will he finish the race? A marathon is 26.2 miles long.

Solution. The problem states (twice) that we assume that Masatake runs at a constant rate. Therefore the relation between minutes and miles is linear and the rate of change is constant. Note that the order of the variables is not specified, so that we can talk about the rate as minutes per mile or miles per minute. As we'll see, the context often shows us which to choose: in this case since the information desired has to do with time, we probably should describe the rate as minutes per mile. In any case, it is desirable to be flexible in the move from one to the other. Using the two measurements we can find that rate: between the two sightings he runs 3 miles in 21 minutes, so is running at a rate of 21/3 = 7 minutes per mile. When Toby saw Masatake, he still had 18.2 miles to run. At 7 min/mi, remembering that

$$\text{Miinutes} = \frac{\text{Minutes}}{\text{Miles}} \times \text{Miles},$$

it will take him (7)(18.2) = 127.4 minutes, or about 2 hours and 7 minutes more. So Toby expects him to finish the race at 11:08 AM. If we also ask, at what time did Masatake start the race? - we know that he started 5 miles before he passed Jaime, and that is $5 \cdot 7 = 35$ minutes. Masatake started the race at 8:05, and he will have run the whole marathon in 3 hours and 3 minutes.

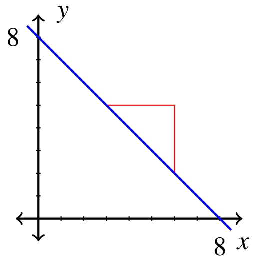
Notice that in working this problem we did not seek an equation to solve, but instead thought about the problem algebraically, using the basic rate equation. The basic fact used here is that the time running between two points is proportional to the distance between the points, and we find the constant of proportionality, 7, using the two given points. In the end, since we now know the initial value of time, we can write the equation for the line:

$$Time = 8:05 + 7\ (Miles),$$

however, be careful in computing with these numbers, since there are 60 minutes (not 100, as the notation might suggest) in an hour.

Example 2.

Given the point $P : (3, 5)$ and the number $m = -1$, find the equation of the line through through P with slope m.



Solution. Following the above, the point $X : (x, y)$ is on the line if the slope calculation with the points X and P gives -1:

$$\frac{y-5}{x-3} = -1$$

This simplifies to $y = -x + 8$.

Example 3.

Given the points $(2, 1), (-1, 10)$, find the equation of the line through those points. First we calculate the slope using the given points:

$$\frac{10-1}{-1-2} = \frac{9}{-3} = -3$$

Now, the equation of the line is given by the slope calculation using the generic point (x, y) and one of the given points (say (2,1)):

$$\frac{y-1}{x-2} = -3$$

or, $y - 1 = (-3)(x - 2)$, which simplifies to $y = -3x + 7$.

Example 4.

Given a straight line on the coordinate plane, such as that in figure 2, find its equation. One way to do this is to discover the values of b and m by locating the y intercept and drawing a slope triangle. In figure 2,the y-intercept is 7, and if we go across by 1 unit, the graph goes up by 3 (be careful to note the different scales on the coordinate axes. This gives $y = 3x + 7$ as the graph. Another way is to locate two convenient points (such as (1,10) and (6,25) since the line goes through these intersections of gridlines), and use them to calculate the slope.

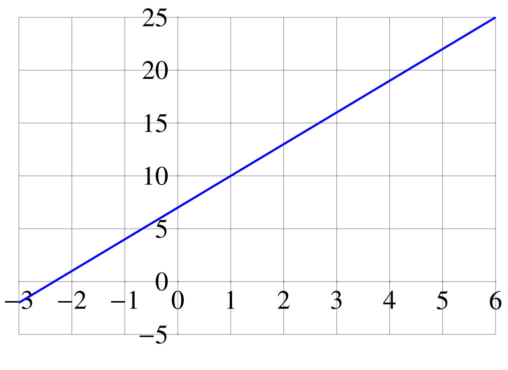


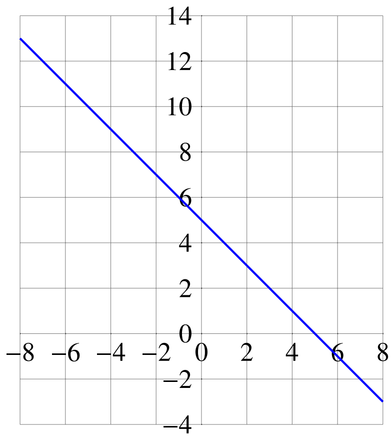
Figure 2

Typically, the first thing to do in trying to understand a relation is to make a table of solutions to see what information we can gather. Then, plot the data points on a graph, and connect them. As an example, consider the relation $x + y = 5$. Make a table of representative values.

x	-8	0	1	3	4	5	8
y	13	5	4	2	1	0	-3

From the table we see that as x increases, y decreases. In fact, whenever x increases by 1, y decreases by 1, confirming that the slope is -1

Here is the graph:



Connecting the points on the graph, we get a straight line. Every point on the graph is a solution, even if it wasn't included in our list of solutions. For any point (x, y) on the graph, if we add x and y we get 5.

Notice that if we shift the line down by 5 units (or to the left by 5 units), the slope is still -1, but the new line goes through the origin. This new line then is the graph of the equation $y = -x$, which can be rewritten as the relation $x + y = 0$. Another way to see this is to notice that moving a point downward by 5 units is the same as subtracting 5 from the y coordinate of a point (with no change in the x coordinate). We can write this as $(x, y) \rightarrow (x, y - 5)$, and say: under the downward shift by 5, the point (x, y) goes to $(x, y - 5)$. We can write this $x_{\text{new}} = x$, $y_{\text{new}} = y - 5$, so the relation $x + y = 5$, rewritten as $x + (y - 5) = 0$ becomes $x_{\text{new}} + y_{\text{new}} = 0$.

We will return to the subject of shifts later in this chapter; for now it suffices to note that a the geometric act of shifting downward by 5 units amounts to the algebraic act of replacing $y - 5$ by y. Similarly, a shift to the left by 5 units is realized algebraically by replacing $x - 5$ by x.

Example 5.

Shift the line $y = x$ upwards by two units, so that the point (x, x) goes to $(x, x + 2)$. In particular, notice the equation of the new line is $y = x + 2$. Similarly, a shift downward of 3 units of the graph of $y = 2x$ brings us to the graph of the equation $y = 2x - 3$.

Let's return to the signal characteristic of a linear relation between the variables x and y: the rate of change of y with respect to x is constant, and this constant is the slope of the graph of the relation. In particular, if a situation is given in which two variables are related and there is a constant rate of change, then the relation is linear. What we have wanted to observe in the last few examples is that if we shift the graph of a linear function so that the new graph goes through the origin, then the new graph is a graph of a proportional relation. Furthermore, the rate of change of y with respect to x along both graphs is the same.

Compare properties of functions (linear) presented in a different way (algebraically, graphically, numerically in tables or by verbal descriptions). 8.F.2.

Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or table of values. 8.F.4.

Up until now we have been speaking informally about a "relation" between the variables x and y rather than "function" and now is a good time to put these words on a firmer footing. Technically speaking, a *relation* between the variables x and y is a region in the plane. Examples:

a. $x < y$, the relation x is less than y, is the region in the plane below the line through the origin that makes a 45° angle with the x-axis;

b. x and y are both less than 1 and greater than 0 is represented by the unit square in the first quadrant;

c. a line in the plane expresses the relation "(x, y) is a point on the line'." $5x - 3y + 7 = 0$ is a relation expressed algebraically: it consists of all pairs (x, y) that satisfy this relation.

In this chapter, we will focus on *linear relations*. A *linear relation* is expressed by a line in the plane: (x, y) are in this relation precisely when (x, y) is on the line. As we learned in the preceding section, if the line is non-vertical, we can describe it by an equation of the form $y = mx + b$. If the line is vertical, there is no relation in the colloquial sense, because x is always the same number, while y can be anything. Similarly, if the line is horizontal ($y = 0x + b$), there is no discernible relation since x can be anything, and y always remains equal to b.

When the relation is given as a recipe for going from a value of x to its related value y, then we say that y is a *function* of x. We look for functions when we have a situation involving two variables x and y, and we have a strong suspicion that the value of x somehow determines the value of y. Then we look for the recipe that makes the "somehow" explicit. For example:

a. Our car salesman's monthly salary is determined by the number of cars sold and indeed we were given the recipe: the monthly salary is \$1000 plus \$250 for each sale in that month. Then we translated this to the algebraic expression: $C = 1000 + 250N$, where C is the number of income for N cars sold. Yes, C, compensation, is a function of the number N of cars sold.

b. The equation $y = mx + b$ expresses a function. The recipe is this: First, pick a number x. Second multiply it by m. Third, add b. The game described in example 12 of chapter 1: "Pick a number from 1 to 20. Add three and double the result. Add 8. Take half of that number. Subtract your original number." It is a complicated way of describing the function: for any x, go to 7, which corresponds to the line $y = 7$, and that is the basis of the "trick:" to every number in the domain, the function assigns the number 7. We will continue and deepen this discussion in Chapter 5.

Example 6.

Consider the relation: the sum of two numbers is 5.

Let x and y represent the two numbers. The sum of x and y is $x + y$; the assertion is that this is 5. This relation can be expressed by the equation

$$x + y = 5$$

The recipe: "Pick a number for x and solve for y" describes a function since we know how to solve linear equations, so this does give us a y for every x. If we write the solution symbolically we get: $y = 5 - x$, whose set of rules are: given the input x, subtract it from 5: that is the output y. This is one reason to think of the function as a black box rather than as a set of instructions: for there could be many different sets of instructions that give rise to the same function (and that is the basis for this kind of "math trick.")

Example 7.

Let's return to Masatake's marathon. His friend Jack also ran the race, but because of the crowd he didn't start running until 8:10. Jack however runs faster than Masatake, at 6.8 minutes per mile. Does Jack finish before Masatake?

Solution. At 6.8 minutes per mile, Jack runs the marathon in (26.2)(6.8) = 178.16 minutes, or 2 hours and about 58 minutes, and arrives at the finish line at 11:03 - a photo finish with Masatake!

To summarize: to find the equation of the line through the two points (x_0, y_0), (x_1, y_1), first calculate the slope of the line using the given points:

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

If (x, y) is any point in the plane then it is on the line L if and only if the slope calculation gives m:

$$\frac{y - y_0}{x - x_0} = m$$

If the variables x, y are in a linear relation (that is, the graph of the relation is a line), the relation can be expressed in the form of the function $y = mx + b$, where m is the slope and b is the y-intercept.

A *linear relation*, as we have been using it, is determined by an equation of the form $Ax + By = C$. If both A and B are zero, we just have the statement $0 = C$, which is not describing any relation between x and y. If just $B = 0$, we get the vertical line $x = C/A$ which does not describe a function. If just $A = 0$, we get the line $y = C/B$ which describes the constant function: for any x, let $y = C/B$. If A and B are both nonzero, we can write the equation in the form $y = mx + b$, exhibiting y as a function of x: $y = -(A/B)x + (C/B)$. The graph of this relation is a non-vertical, non-horizontal line of slope $-(A/B)$ and y-intercept C/B.

Example 8.

Find the slope of the line given by the relation $3x - 7y = 11$, and write the equation in slope-intercept form.

Solution. By the above, the slope of the line is $-(3)/(-7) = 3/7$. Set $y = 1$ and solve the equation for x. We find $x = 6$, so (6,1) is a point on the line. We could also use the laws of arithmetic to write y in terms of x, getting

$$y = \frac{3}{7}x - \frac{11}{7}$$

from which we conclude that the slope of the line is 3/7, and the point $(0, -11/7)$ is on the line.

Section 3.2 Parallel and Perpendicular lines

Example 9.

Let us return to the water measurement activity in example 3 of chapter 2, and plot both sets of weight data as the y coordinate, and the height on the x-axis. Connecting the two sets of data with a line, we get the following graph, where the blue (upper line) is the measured weight and the red (lower line) is the weight of the water

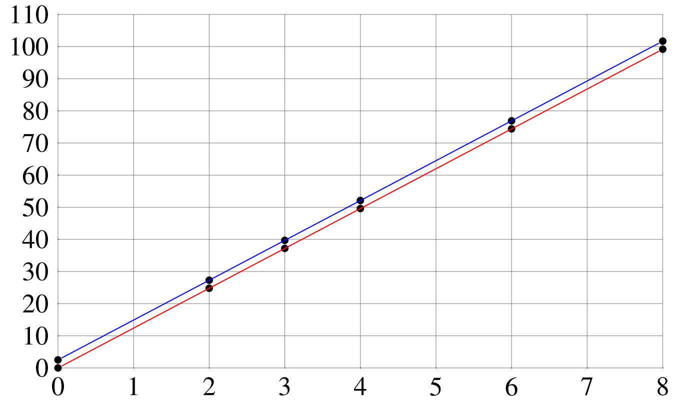


Figure 4

In many of the examples discussed above, we talked about changes in the relation between the two variables that result from a shift, or a translation. To make this precise: a translation by (a, b) is a motion of the plane by a horizontally, followed by a motion of the plane by b vertically.

Example 10.

Consider the graph of $x + y = 5$, as described in Example 6. Shift the graph upward by 3 (that is by $(0, 3)$). What is the equation of the new line?

Solution. Since we have increased y by 3, $x + y$ has increased by 3, so the equation of the new line $x + y = 8$. Here is another way to see this. Let $(x_{\text{new}}, y_{\text{new}})$ be the coordinates of the point to which $(x_{\text{old}}, y_{\text{old}})$ is moved. We know that

$$x_{\text{old}} = x_{\text{new}} \quad \text{and} \quad y_{\text{old}} = y_{\text{new}} - 3$$

The equation of the old line is

$$x_{\text{old}} + y_{\text{old}} = 5$$

which is, in terms of the new coordinates:

$$x_{\text{new}} + y_{\text{new}} - 3 = 5$$

Since we are drawing the lines on the same coordinate plane, we can remove the word "new" to get

$$x + y - 3 = 5 \quad \text{or} \quad x + y = 8$$

Finally we can verify these observations with the table:

	x	−8	0	1	3	4	5	8
Old	y	13	5	4	2	1	0	−3
New	y	16	8	7	5	4	3	0

EXAMPLE 11.

Translate the line $y = 2x$ by one unit in each coordinate, so that $(x, 2x)$ goes to $(x+1, 2x+1)$. Find the equation of the new line.

SOLUTION. Here we have the relation

$$x_{\text{old}} = x_{\text{new}} - 1 \quad , \quad y_{\text{old}} = y_{\text{new}} - 1$$

So the equation

$$y_{\text{old}} = 2x_{\text{old}}$$

becomes

$$y_{\text{new}} - 1 = 2(x_{\text{new}} - 1)$$

removing the word "new," this becomes $y - 1 = 2(x - 1)$, which simplifies to $y = 2x - 1$

> If the line $y = mx$ is translated by (a, b), then the equation of its image is
>
> $$y - b = m(x - a)$$

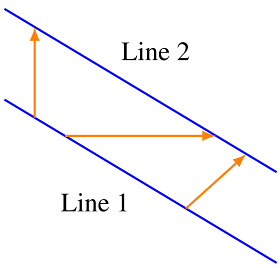


Figure 5

Now, two lines are said to be *parallel* if there is a translation that takes one into the other. The statement in the box tells us that if two lines are parallel, they have the same slope. It is also true that if two lines have the same slope, they are parallel, that is: there is a translation that takes one into the other. How do we find that translation? Consider the image in Figure 5 depicting two lines with the same slope:

Each orange arrow represents a translation. Observe, using two pieces of transparent graph paper, that each of those translations takes line 1 to line 2. You should conclude that, given any points P on line 1, and Q on line 2, the translation from P to Q takes line 1 to line 2. Since this diagram can be of any two lines with the same slope (we have omitted coordinates to emphasize this point); we can conclude

> Two lines are parallel if and only if there is a translation of one line to the other. Parallel lines have the same slope and lines with the same slope are parallel.

In chapter 8 where we will study translations in more detail, we will note that if two lines are parallel they never intersect, and conversely, if two lines never intersect they are parallel. This statement (a version of the *Parallel Postulate* of Euclid) cannot be verified by observation because we cannot see infinitely far away. For this reason, it has been discussed throughout history, the issue being whether or not it is a necessary part of planar geometry. It turned out, in the 19th century, that it is, for there are geometries different from planar that satisfy all of the conditions of planar geometry but the Parallel Postulate.

Two lines are *perpendicular* if they intersect, and all angles formed at the intersection are equal. This of course is the same as saying that all these angles have measure 90°.

To understand perpendicularity, we will need the idea of *rotation*. A rotation is a motion of the plane around a point, called the center of the rotation. To visualize what a rotation is, take two pieces of transparent coordinate paper, put one on top of the other and stick a pin through both piece of paper. The point where the pin intersects the paper is the *center* of the rotation. Now any motion of the top piece of paper is a visualization of a rotation. For any figure on the bottom piece of paper, copy it onto the top, then rotate the top piece of paper and copy the figure on the top to the bottom. That image is the rotated image of the original figure.

In figure 6, we see the result of rotating the red line (the line with positive slope) through a right angle (90°) with the center C. The blue line (with negative slope) is the image of the red line under the rotation.

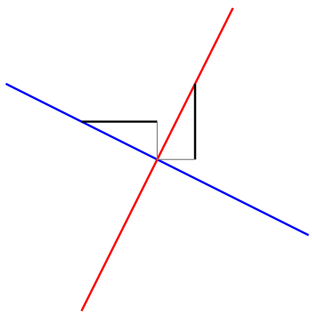


Figure 6

Notice that the dark lines and the light lines correspond under the rotation, so they have the same lengths. Notice also that these are the triangles that are drawn for the slope computation except that the rise and run has been interchanged: in terms of lengths, rise(red) = run(blue), run(red) = rise(blue). However, there is one last (and important) thing to notice: the slope computation is in terms of differences between coordinates, and not lengths. In our diagram the sign of one pair of differences (represented by the black lines) has changed, while the sign of the other pair of differences has not. We can summarize this as follows:

For the red line,

$$\text{slope(red)} = \frac{\text{length(black)}}{\text{length(gray)}}$$

and for the blue line,

$$\text{slope(blue)} = \frac{\text{length(gray)}}{-\text{length(black)}}$$

from which we can conclude that the product of the slopes of the blue and red lines is -1. Since we did not use any coordinates to make this argument, this statement is general, so long as neither line is horizontal or vertical.

To recapitulate: if we rotate a line L (red in figure 6), (90°) about a point P on the line, getting the new line L' then the products of the slopes of L and L' is -1. The following statement follows from this assertion:

> If lines $L1$ and $L2$ are perpendicular at their point of intersection, then the product of their slopes is -1. If the product of the slopes of lines $L1$ and $L2$ is -1, then they are perpendicular at their point of intersection.

To see this, first, let us suppose two lines L_1 and L_2 intersect perpendicularly at a point P. Now rotate the line L_1 by 90 ° to get the line L_1'; then the product of the slopes of L_1 and L_1' is -1. But since there is only one line perpendicular to L_1 at P, L_1' and L_2 are the same line. To show the second statement: again, suppose that L_1 and $L-2$ intersect at P but this time suppose the products of their slopes is -1. Again rotate L_1 by 90 ° to get the line L_1', and again L_1' has the same slope as L_2, so must coincide with L_2.Thus L_1 and L_2 are perpendicular.

When the lines are given by a linear relation, it is easy to write the relation of the line perpendicular to it:

> If a line L is given by the relation $Ax+By = C$, then the equation $Bx-Ay = D$ (for any D) describes a line L' perpendicular to L.

This is because line L has slope $-A/B$, and line L' has slope B/A.

EXAMPLE 12.

Consider the line L given by the equation $3x + 4y = 20$. The last statement tells us that all lines perpendicular to L have an equation of the form $4x - 3y = D$, where D is to be determined by the point of intersection of the two lines. So, to find the equation of the line L' perpendicular to L that passes through the point $(4, 2)$, we just calculate $4(4) - 3(2) = 10$; the equation of L' is $4x - 3y = 10$.

Chapter 4
Simultaneous Linear Equations

Section 4.1: Understanding Solutions of Simultaneous Linear Equations

Analyze and solve pairs of simultaneous linear equations. Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously. 8.EE.8a

Simultaneous linear equations refers to a pair of equations of the form $Ax + By = C$, where A, B, C are specific numbers, positive or negative. To say they are simultaneous is to ask: for what, if any, values substituted for the variables (x and y) are the equations both true at the same time? Those pairs of values are the *solutions* of the simultaneous equations. To illustrate: $x + 2y = 10$, $x - 3y = 0$ is a pair of equations, describing two relations between the variables x and y. If the context requires them to both be true, they are simultaneous. A solution is $x = 6$, $y = 2$, because that substitution makes both statements true. In this chapter, we want to explore procedures, both algebraic and graphical, to determine the solutions of simultaneous linear equations.

Example 1.

$x = 21$, $y = 17$ is a pair of simultaneous linear equations. Clearly there is only one solution, namely $x = 21$, $y = 17$. Suppose I want to disguise this: these are the ages of my older siblings, and I am asked for their ages. OK, I'll say: the sum of their ages is 38. The inquisitor is not satisfied: there are many pairs of numbers whose sum is 38. OK, I add the information that the difference in their ages is 4. Now, can the inquisitor determine their ages? If he lets them be x and y, he can now write down the two pieces of information algebraically: "the sum of their ages is 38" becomes $x + y = 38$. "The difference in their ages is 4" becomes $x - y = 4$. The two equations

$$x + y = 38, \quad x - y = 4$$

are a pair of simultaneous linear equations, for which the actual ages of my siblings are a solution. But is this enough information for the inquisitor to find the solution? If I know the sum and difference of two numbers, do we know the numbers? The answer to this is "yes," and to solve the problem we must discover how to get from the sum and difference of two numbers back to the originals.

Example 2.

Let's give another example, in the form of a game, that illustrates the same process:

1. Pick two numbers.
2. Double one and add the other. Tell me the result.
3. Now exchange the numbers and do the same. Tell me the result.

The numbers I received are 30 and 27. After a second, I say "Your two numbers were 8 and 11."

How did I get the answer so fast? Let's investigate this by analyzing the process. In Examples 1 and 2, we start with a particular pair of numbers. Then we perform algebraic operations on them: in the first problem we added the two numbers and then subtracted the two numbers. From $x = 21$, $y = 17$ we went to $x + y = 38$, $x - y = 4$. The challenge to the inquisitor is to find a way to go back. Maybe he should add these two equations:

$$x + y = 38, \quad x - y = 4; \quad \text{adding we get} \quad (x + y) + (x - y) = 38 + 4,$$

which simplifies to $2x = 42$, so we've found x: $x = 21$. Now let's subtract the two equations:

$$x + y = 38, \quad x - y = 4; \quad \text{subtracting we get} \quad (x + y) - (x - y) = 38 - 4,$$

which simplifies to $2y = 34$, so we have also found y: $y = 17$.

Let's analyze the second example in the same way. The two numbers picked are x and y. In step 2; we calculate $2x + y$ and inform the questioner that $2x + y = 30$. Then, in step 3, we form $2y + x$ and assert its value is 27. So the questioner knows that

$$2x + y = 30, \quad 2y + x = 27$$

Let's now do the same with this information: Add what we know:

$$(2x + y) + (2y + x) = 57,$$

which simplifies to $3(x + y) = 57$, or $x + y = 19$. Now subtract what we know:

$$(2x + y) - (2y + x) = 30 - 27$$

or $x - y = 3$. Now we know the sum and differences of the two numbers, so we can apply the technique of eExample 1 to obtain $x = 11$, $y = 8$.

The point of these examples is to see how to get from one pair of simultaneous equations to another, so that the solution set is the same. In one direction, when we want to disguise the numbers, we continue this process until we have sufficiently confounded the subject. In the other direction we select operations that unscramble that information. The tools we employ are these:

a. Add equals to equals ($x = 21$, $y = 17$ became $x + y = 38$);

b. Subtract equals from equals ($x = 21$, $y = 17$ became $x - y = 4$);

c. Multiply equals by a nonzero number (in the case of $2y = 34$,we multiply both sides by $1/2$).

Using these operations we can get from one pair of simultaneous equations to another pair so that the solution for the two pairs of equations does not change. Note that it is important, since we have two unknowns that we must have, at every stage, two equations. If I have in mind the numbers 17 and 21, and I tell you that the sum is 38, you do not have enough information to find the numbers. I either have to tell you that one of the numbers is 17 (or 21), or I have to give you another piece of scrambled information, such as: the difference of the two numbers is 4.

The point is not that we apply these operations at random, but we do so in order to reach our objectives. As we shall see in the next section, it is the form of our given information and the actual known numbers that show us the operations to use. There is one last tool for solving, that of substitution:

d. Replace an expression in one equation by an equal expression obtained from the other equation.

Example 3.

Lovasz has 5 marbles more than twice the number of marbles that Tonio has. Together they have 107 marbles. How many marbles does Lovasz have?

Solution. First we represent the unknown numbers by letters: Let L be the number of marbles that Lovasz has, and T the number of marbles that Tonio has. We are told that "Lovasz has 5 marbles more than twice Tonio's"; this translates to $L = 5 + 2T$. The second fact is that the sum of all the marbles is 107, so $L + T = 107$. The first equation tells us that L and $5 + 2T$ are the same number, so we can replace L by $5 + 2T$ in the second equation to get:

$$5 + 2T + T = 107.$$

From Chapter 1 we know how to solve a linear equation in one variable: combine like terms on the left and subtract 5 from both sides to get $3T = 102$, so $T = 34$: Tonio has 34 marbles. To find L, we turn to the first equation and replace T by 34, since we know these are equal, and we have $L = 5 + 2(34)$, so $L = 73$. We can use the second equation again to check this result: $73 + 34 = 107$.

Before going to the use of these operations to solve systems, we turn to the representation of this process by graphs.

Example 4.

Consider the linear equations $3x + y = 7$, $x + 3y = 5$. Graph both equations on a coordinate plane and find the coordinates (x, y) of the point of intersection.

Solution. The slope of the first line is -3, and of the second, $-1/3$. Since the slopes are not the same, the lines are not parallel, so they must intersect. Now, using a point on each line (for example, the y-intercepts $(0, 7)$ for the first line and $(0, 5/3)$ for the second, we graph the lines as in Figure 1.

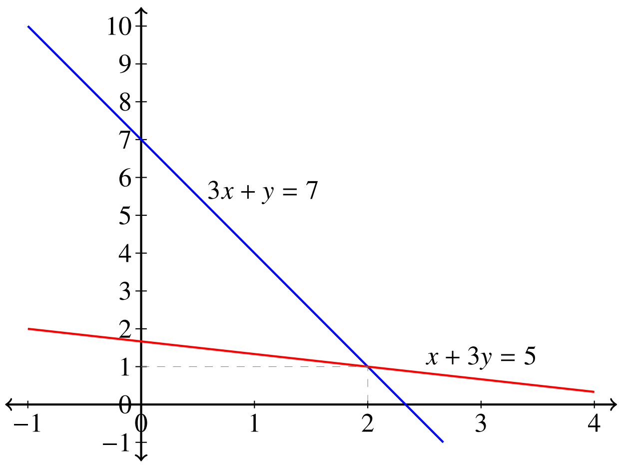


Figure 1

The figure shows the point of intersection to be $(2, 1)$. We should confirm that by checking that the substitution $x = 2$, $y = 1$ satisfies both equations:

$$3(2) + 1 = 7, \quad 2 + 3(1) = 5$$

In this example, we have graphed the given linear relations, and read off, from the graph, the coordinates $(2, 1)$ of the point of intersection. As this point lies on both lines, those coordinates ($x = 2$, $y = 1$) satisfy both equations. This is what it means to "solve the pair of simultaneous equations:"

Example 5.

Consider the linear equations $x - 2y = 8$, $2x + 5y = 34$. Graph each equation on the same grid with the same axes , and read off the coordinates (x, y) of the point of intersection.

Solution. Again we see that the two lines have different slopes (1/2 and −2/5), so the lines are not parallel and have a point of intersection. We draw the graphs of these equations (see Figure 2) and read off the coordinates of the point of intersection as (12, 2). Since (12, 2) lies on both lines, the values $x = 12$, $y = 2$ will satisfy both equations.

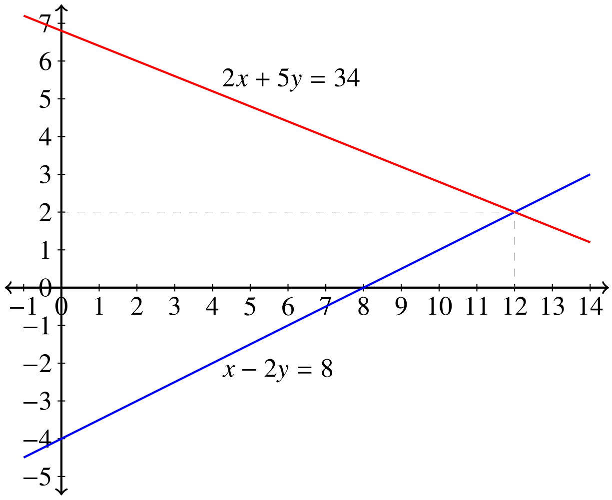


Figure 2

Furthermore, since there is only one point of intersection of two nonparallel lines, this is the unique solution.

What happens if the lines are parallel? We look at this case in the next example:

Example 6.

Consider the linear equations $2x + 5y = 10$, $4x + 10y = 40$. Graph each equation, and look for the point of intersection. These equations provide graphs of two different lines (since they have different y-intercepts, (0, 2) and (0, 4)) that are parallel, since they have the same slope (−2/5). In particular, there is no point of intersection, which tells us that there are no values $x = a$, $y = b$ that satisfy both equations, The graph of these lines (Figure 3) confirms this.

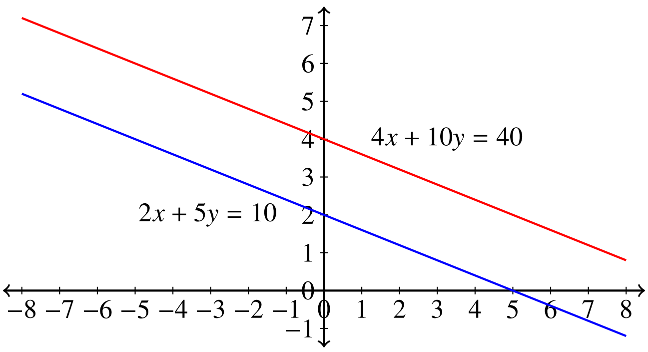


Figure 3

Example 7.

Now, consider the linear equations $2x + 5y = 20$, $4x + 10y = 40$. The graphs of these equations are the same, since the lines they describe have the same slope and the same y-intercept.

In fact, if we put these equations into slope-intercept form, they both simplify to $y = -(\frac{2}{5})x + 4$. In this case, there are infinitely many solutions to the pair of equations, since the coordinates of every point on the line satisfies both equations. Notice that one of the original equations is a multiple of the other (divide both sides of the second equation by 2), so they are equivalent expressions. This will always be the case when a pair of simultaneous linear equations has more than one solution.

Summary: Given a pair of linear equations, there are three possibilities for simultaneous solutions:

1. The rate of change of y with respect to x is different for the two equations. In this case the graphs of the equations are lines with different slopes so are nonparallel, and intersect in a point. The coordinates of this point give the unique solution of the pair of equations.

2. The rate of change of y with respect to x is the same for the two equations, but they have different intercepts. In this case, the equations define lines with the same slope and thus parallel. If the lines are different, there is no solution to the simultaneous equations.

3. The rate of change of y with respect to x is the same for the two equations, and the equations define the same line. In this case, the coordinates of any point on the line gives a solution for the pair of equations.

In short, if two lines have different slopes (are not parallel) then there is a (single) point of intersection. If two lines have the same slope (are parallel), then either there is no solution, or they are the same line and there are many solutions.

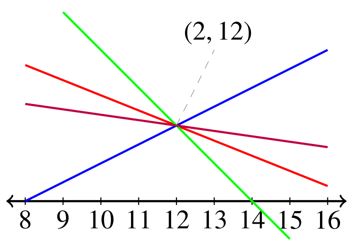


Figure 4

Take a look at the graphical implications of the operations on simultaneous equations described above. Start with the equations of Example 5: $x - 2y = 8$, $2x + 5y = 34$. The graphs are shown in Figure 2. Add the two given equations, getting $3x + 3y = 42$. If we subtract the first from the second, we get $x + 7y = 26$. Put the graphs of these two additional equations onto the Figure 2, resulting in Figure 4. The green and purple lines are the graphs of these new equations. Note that the graphs go through the same point; that is, the solution for the original pair of equations is also the solution for the new set of equations. So, we may now proceed to solve using the new set of equations. Our objective is to manipulate the equations with these operations so that we end with equations whose graphs are horizontal and vertical lines (in the case the graphs of $x = 12$ and $y = 2$).

Section 4.2 Solving Simultaneous Linear Equations Algebraically

Solve systems of two linear equations in two variables algebraically, and estimate solutions by graphing the equations. Solve simple cases by inspection. For example, 3x + 2y = 5 and 3x + 2y = 6 have no solution because 3x + 2y cannot simultaneously be 5 and 6. 8.EE.8b

Method of Substitution

This is a straightforward method, and is to be used when one of the pair of simultaneous linear equations expresses one variable (say y) in terms of the other (say x) . Then we can replace the y in the other equation by the expression in x, and obtain a new equation with only one variable. Example 3 above illustrates this method; here we further develop the idea.

Example 8.

I went to the grocery store and bought 15 pounds of grapefruit and oranges. I bought 3 fewer pounds of grapefruit than oranges. How many pounds of each fruit did I buy?

Solution. Pick letters to represent the number of pounds of grapefruit and oranges bought. I'll pick G for grapefruit, and R for oranges (since O is not a convenient symbol, being so close to the symbol for zero). The first statement says that $G + R = 15$. The second statement says that $G = R - 3$. So we can replace G in the first equation by $R-3$, giving us $R-3+R = 15$. This simplifies to $2R = 18$, so I bought 9 pounds of oranges. Using the second statement, we see that I bought 6 pounds of grapefruit.

Example 9.

Two numbers (x, y) are related by the equation $5x-y = 17$. We also know that y is a function of x, given by $y = 2x + 4$. Find the solution.

Solution. Substituting, we get $5x - (2x + 4) = 17$. This we can solve for x: the equation becomes $3x - 4 = 17$, leading to $3x = 21$, and so $x = 7$. Now we use the functional relation between y and x to find y; $y = 2(7) + 4$, so $y = 18$.

Example 10.

Sometimes a little work is needed to write one of the variables in terms of the other. For example, solve

$$2x + y = 12$$
$$2x - 3y = 4$$

Solution. We can rewrite the first equation as $y = 12 - 2x$, and then replace y in the second by this expression:

$$2x - 3(12 - 2x) = 4$$

Now we solve this equation for x, first simplifying to get $2x - 36 + 6x = 4$, or $8x = 40$, so that $x = 5$. The corresponding y is $y = 12 - 2(5) = 2$. Note another way to solve the equation: subtract the second equation from the first to get $4y = 8$, and thus $y = 2$.

Example 11.

Another day I went to the store and spent \$26.25 for 15 pounds of grapefruits and oranges. The grapefruit cost \$1.25 per pound and the oranges cost \$2.00 per pound. How many grapefruits and oranges did I buy?

SOLUTION. Using the same letters as in Example 7, the first equation tells us that $G+R = 15$. G pounds of grapefruit cost me \$1.25(G), and R pounds of oranges cost me \$2.00(R). The sum is \$26.25, giving me the equation

$$1.25G + 2R = 26.25$$

The first relation tells me that $R = 15 - G$, so I can replace R in the second equation by $15 - G$, giving me

$$1.25G + 2(15 - G) = 26.25$$

This becomes $1.25G + 30 - 2G = 26.25$, which simplifies to $.75G = 3.75$, so $G = 5$. Then, returning to the relation $G + R = 15$, we see that $R = 10$. I bought 5 pounds of grapefruit and 10 pounds of oranges.

Some of the choices made in this problem were arbitrary: to begin with, We could have solved for G in terms of R ($G = 15 - R$), and then written the equation in R:

$$1.25(15 - R) + 2R = 26.25$$

but it is better to have to multiply 2 and 15 instead of 1.25 and 15. Nevertheless the result would have been the same. And again, when we came to $.75G = 3.75$; we could change to fractions to get

$$\frac{3}{4}G = \frac{15}{4}$$

from which we see directly that $G = 5$. In almost all cases such choices have to be made, and should be made on the basis of making one's work as simple as possible.

The Substitution Algorithm is

1. Rewrite one of the equation so as to express one variable in term of the other;
2. Substitute that expression for that variable in the other equation;
3. Solve for the second variable, and put that value in the first equation to find the solution.

Method of Elimination

For many pairs of linear equations to be solved simultaneously, there are ways to find the solutions that are easier than trying to express one variable in terms of the other. Here we will describe a method that is the default method for most computational programs.

EXAMPLE 12.

Solve

$$3x + 2y = 12$$
$$2x + 2y = 10$$

SOLUTION. If we look carefully at the equations, we see that, on the left side, the difference between the first and the second expressions is x, and the difference on the right side is 2. Since we can add and subtract equations without changing the solution set, we thus choose to subtract the second equation from the first to get $x = 2$. Putting that value in either equation gives us an equation in the single unknown y, and we conclude from either that $y = 3$.

Example 13.

Solve $6x + 2y = 20$, $3x - y = 2$.

Solution. From the previous example, we learned that if the coefficient of y is the same, we can combine the equations so as to eliminate y, then solve for x, and use that known value for x in one of the preceding equations to solve for y. We can arrange this by multiplying both sides of the second equation by 2, giving us the pair of equations

$$6x + 2y = 20$$
$$6x - 2y = 4$$

If we add these equations y subtracts put and we get $12x = 24$, from which we find that $x = 2$. Now putting this value of x into either equation, gives us $y = 4$, so the complete solution is $(2, 4)$.

We could also subtract the equations to get $4y = 16$, giving us the value $y = 4$; upon substitution of this value in either equation, we get $x = 2$.

Example 14.

Solve

$$\frac{3}{5}x + \frac{1}{3}y = 12$$
$$\frac{1}{3}x + \frac{1}{2}y = 18$$

We include this example to illustrate that the complexity of the numbers involved should not be a deterrent to applying these methods, but one must be careful. There are various ways of solving this problem (besides feeding the data into a computer program) that make the arithmetic manageable. First, we might clear of fractions by multiplying the first equation by 15 and the second by 6 to get:

$$9x + 5y = 180$$
$$2x + 3y = 48$$

Now multiply the first equation by 3 and the second by 5 so as to get the coefficients of y the same:

$$27x + 15y = 540$$
$$10x + 15y = 240$$

Subtracting the second from the first produces $17x = 300$, so that $x = 300/17$. We can now put this value of x in any of the preceding equations to solve for y. Let's use the equation $2x + 3y = 48$. We get

$$2\left(\frac{300}{17}\right) + 3y = 48$$

leading to

$$3y = \frac{48 \times 17 - 600}{17}$$

or $3y = 216/17$, giving us the value $72/17$ for y.

We can avoid the large multiplications and divisions by first noting that if we multiply the original first equation by 3/2, we make the coefficients of y the same:

$$\frac{9}{10}x + \frac{1}{2}y = 18$$
$$\frac{1}{3}x + \frac{1}{2}y = 8$$

Now subtract the second from the first to eliminate y and get

$$\left(\frac{9}{10} - \frac{1}{3}\right)x = 10$$

Multiply by 30 to clear of fractions to get $(27 - 10)x = 300$, which is not the same equation for x we have above, so proceed in the same way.

One may be tempted to switch to decimals to get

$$0.6x + 0.33y = 12$$
$$0.33x + 0.5y = 8$$

and then use a calculator for the computations to follow. However, one has to be careful: 0.33 is not 1/3, but an approximation of 1/3. As one goes through calculations, the error in the approximation tends to grow, often to the extent to make the end result untrustworthy. So, for example, to achieve accuracy within 2 decimal points, one should start with the approximation 0.3333 for 1/3, making the calculations that much more difficult.

We summarize this procedure in the following set of rules. Keep in mind that in many of the examples above, and in the problems for discussion and homework, the particular numerical coefficients give a clue on how to proceed. In real-life problems we do not have that luxury: experimentally determined numerical coefficients are hardly ever so convenient.

The Elimination Algorithm is

1. Multiply the equations by nonzero numbers so that the coefficients of one of the unknowns are the same;
2. Take the difference (or sum) of the equations to obtain a new equation in just one unknown;
3. Solve for that unknown, then substitute that value in one of the original equations to solve for the other unknown.

Comments:

1. The first step is to arrange for the coefficients in the two equations of one of the variables to be the same. There can be many ways of doing this; they are all valid. For example, for the pair

$$2x + 4y = 12.30$$
$$x + 5y = 13.20$$

we could have multiplied the first equation by 5 and the second by 4 to obtain:

$$10x + 20y = 61.50$$
$$4x + 20y = 52.80$$

Now the difference leads to $6x = 8.70$, and $x = 1.45$. It worked; nevertheless, a good rule to follow is this: look for the simplest multipliers to use. Here it would have been to multiply the second equation by 2 so as to eliminate x.

2. The second step suggests that the elimination may involve taking the sum, rather than the difference. For example:

$$2x + 6y = 38$$
$$x - 3y = 11$$

Step 1 suggests multiplying the second equation by 2 to obtain

$$2x + 6y = 38$$
$$2x - 6y = 22$$

Now, adding the equations will eliminate y and we get $4x = 60$, so $x = 15$. Notice that, if we took the difference, we get $12y = 16$, so $y = 4/3$.

3. When we eliminate one of the variables, suppose both disappear? Consider the pair of equations:

$$2x + y = 7$$
$$4x + 2y = 4$$

Following the algorithm, we multiply the first equation by 2 to get:

$$4x + 2y = 14$$
$$4x + 2y = 4$$

Subtracting the second equation from the first gives the equation $0x + 0y = 10$. Since there are no values of x and y that can make that statement true, the same is true for the original pair: there are no solutions. Notice that the slope of both lines is $-1/2$; that is, the lines are parallel. So, this corresponds to the graphical situation where the lines never cross. Similarly, if we consider this pair of equations:

$$2x + y = 7$$
$$4x + 2y = 14$$

and follow the rules, we end up with $0 = 0$, which is a true statement, but not a very informative one. What we are observing is that both equations define the same line, since the second equation is just double the first.

4. The choice of method to use is up to the solver, and depends upon the coefficients of the equations, just pick the method that is easier. For example, consider the pair of equations

$$3x + 7y = 18, \quad y = 6 + 3x$$

If we apply the elimination method, we first have to rewrite the second equation as $-3x + y = 6$, and then proceed. But, why? Our procedure is to isolate one of the variables and the second equation has done that for us. So, we can go right into the the replacement step, and substitute the value of y in terms of x given by the second equation, into the first, to get

$$3x + 7(6 + 3x) = 18$$

which simplifies to $24x+42 = 18$, leading to $24x = -24$, or $x = -1$. Substituting that value into the other equation gives us $-3 + 7y = 18$, or $y = 3$.

Example 15.

Solve the pair of equations:

$$\begin{aligned} x + y &= 10 \\ 11x + 8y &= 92 \end{aligned}$$

The first equation tells us that $y = 10 - x$; substituting that in the second gives

$$11x + 8(10 - x) = 92$$

which we can now solve for x. We get $11x + 80 - 8x = 92$, which simplifies to $3x = 12$, with the result that $x = 4$. Now substitute that in the first equation to find that $y = 6$. Try the method of elimination, to compare the difficulty of both methods.

Example 16.

In the next example, it is not so clear at first which is the more direct method:

$$\begin{aligned} 4x + y &= 10 \\ 13x + 11y &= 79 \end{aligned}$$

Looking at the two equations, the method of elimination suggests multiplying the first equation by 11, leading to rather large and unwieldy numbers to work with. On the other hand, the first equation easily transforms into the equation $y = 10 - 4x$. Substituting $10 - 4x$ for y in the second equation gives us

$$\begin{aligned} 13x + 11(10 - 4x) &= 79 \\ 13x + 110 - 44x &= 79 \\ -31x &= -31 \end{aligned}$$

from which we get $x = 1$. Now substituting that in the first equation gives us $4 + y = 10$, or $y = 6$.

Don't forget that you have to use both equations! So, for example, if you solve the first equation for y in terms of x, substitute that expression in the *second* equation, not the first. If you substitute in the first, you get a true, but not very useful equation.

Example 17.

In many problems the coefficients will not work out well (as in Example 14 above), and we may be satisfied with an estimate of the solution. In such cases it is advisable to turn to graphical solutions. In fact, graphical solutions can only provide approximate solutions, unless the point in question is at the intersection of gridlines. Furthermore, the estimate is only as good as the grid lines of fine. Let's work an example for which we will be satisfied with an estimate within one decimal point. The equations are:

$$\frac{3}{4}x + \frac{4}{5}y = 16, \qquad \frac{1}{10}x + \frac{7}{10}y = 7$$

In Figure 5 we have graphed the lines corresponding to these equations, from which we can read the approximate solution $x = 12.6, y = 8.2$.

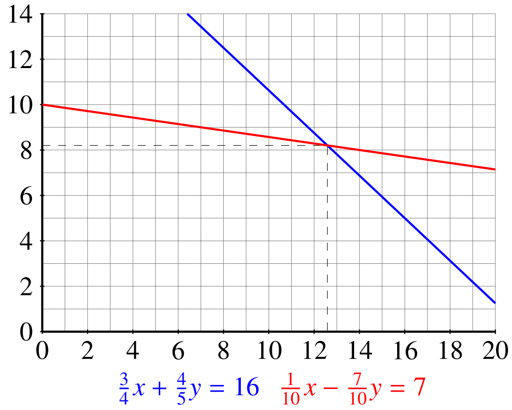


Figure 5

Example 18.

It is very helpful to use a spreadsheet, like Excel, to find graphical solutions. On the spreadsheet, we can calculate two points on each line, and then graph the lines on the same grid. For the pair of equations

$$6x + 5y = 31$$
$$5x - 3y = 0$$

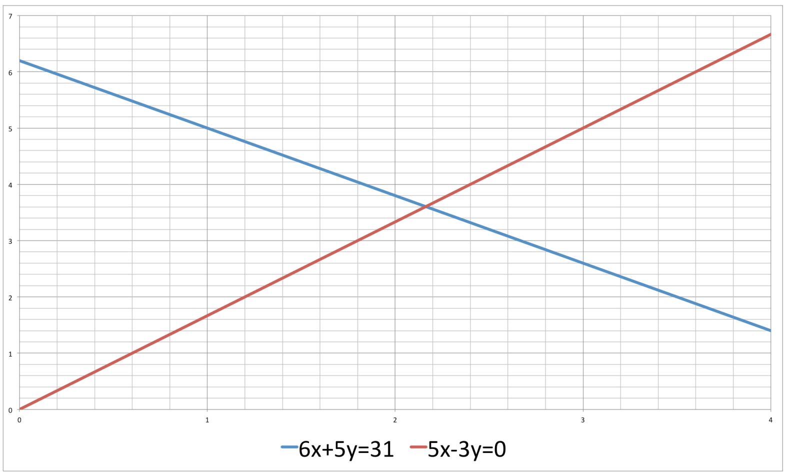


Figure 6

From the graph (see Figure 6) we estimate the coordinates of the point of intersection to be (2.2, 3.6).

Solving Real World Problems using Systems

Solve real-world and mathematical problems leading to two linear equations in two variables. 8.EE.8c

Example 19.

Let's return to grapefruits and oranges (Example 11). Joanne and Rudy shop at the same store. Joanne bought 6 lbs of grapefruit and 4 lbs of oranges, and spent \$8.22. Rudy bought 6 lbs of grapefruit and 5 lbs of oranges and spent \$9.09. What is the cost of a pound of oranges? How much is a pound of grapefruit?

It is always a good idea to study the problem carefully, looking for clues for solving. For example, in this case we see that Rudy bought the same amount of grapefruit as Joanne, and one more pound of oranges than Joanne, and spent \$.87 more than Joanne. So, we conclude that a pound of oranges costs \$.87.

Now, Rudy spent \$9.09, of which $5 \times .87 = 4.35$ was on oranges. Thus he spent $9.09 - 4.35 = 4.74$ on 6 lbs of grapefruit , so each pound of grapefruit is worth $(4.74)/6 = .79$.

In general we may not be so lucky as to see a shortcut right away, so it is always a good idea to apply the rules for solving equations developed in Chapter 1. First: what do we want to find out? - the answer is the cost of a pound of grapefruit, and the cost of a pound of oranges. Let's call those unknowns g and r. (It might be good to point out that these are not the same G and R as in example 11: there the issue was of "pounds of fruit," here it is of "cost of fruit." We have chosen lower case letters to avoid this confusion). Now look at Joanne's purchase: 6 lbs of grapefruit at G per pound, and 4 lbs of oranges at R per pound. This totals to \$8.22, giving us the equation:

$$6g + 4r = 8.22$$

Do the same with Rudy's purchase, to get $6g + 5r = 9.09$. Using the general rules of elimination developed above, we subtract the first equation from the second to get $g = .87$, and continuing as above, we'll find $r = .79$.

Example 20.

Alfredo and Juanita also shop at a store in the same chain, but one that is eight states away. Joanne bought 6 lbs of grapefruit and 3 lbs of oranges, and spent \$10.77. Alfredo bought 3 pounds of grapefruit and 2 pounds of oranges and spent \$5.94. What is the cost of a pound of oranges? What is the cost of a pound of grapefruit?

Here we can't just look at the difference in the purchases: it tells us 3 lbs of grapefruit and 1 lb of oranges costs \$4.83, which is not of much help. But if we double Alfredo's purchase, we can say that 6 lbs of grapefruit and 4 lbs of oranges costs $2 \times 5.94 = 11.88$. This gives us the pair of equations (using the same meaning of g and r as in the preceding example), but eight states away:

$$6g + 3r = 10.77$$
$$6g + 4r = 11.88$$

Now the difference on one side is 1 pound of oranges, and on the other, \$1.11. The cost of one pound of oranges is \$1.11. Let's go back to one of the original equations to find the cost of grapefruit. Alfredo spend \$2.22 on the 2 lbs of oranges, and $\$5.94 - 2.22 = 3.27$ on 3 lbs of grapefruit . Thus one pound of grapefruit costs $3.27/3 = 1.09$.

EXAMPLE 21.

Each day, ferry companies A and B cross the straights of Gibraltar connecting the Spanish port of Tarifa with Tangier and Ceuta. Company A makes 4 round trips to Tangier and 3 to Ceuta, logging 332 miles. Company B makes 2 round trips to Tangier and 4 to Ceuta, logging 302 miles. What are the distances of Tarifa from Tangier and Tarifa from Ceuta?

SOLUTION. Let T represent the distance in miles of Tarifa from Tangier, and C, the distance between Tarifa and Ceuta. Then a round trip from Tarifa to Tangier is $2T$ miles and a round trip from Tarifa to Ceuta is $2C$ miles. Company A makes 4 round trips to Tangier - that logs $4(2T)$ miles - and makes 3 round trips to Ceuta -that logs $3(2C)$ miles. The sum of the distances of all these trips is 332 miles, giving us the equation

$$4(2T) + 3(2C) = 332$$

Applying the same reasoning for company B leads to the equation

$$2(2T) + 4(2C) = 302$$

So, our task is to find the values of T and C that simultaneously satisfies the two equations

$$8T + 6C = 332, \quad 4T + 8C = 302$$

Now solve, to find the answers: Tarifa is 21.1 miles from Tangier, and Ceuta is 27.2 miles from Tangier.

EXAMPLE 22.

Lisa is interested in discovering the rate of gas consumption of her new car, both in city miles and freeway miles. She selects two weeks in which she is driving in the city during the weekdays, and goes on a road trip on the weekend. The following table shows the miles she logged:

	City	Freeway
Week 1	131	210
Week 2	180	120

In each week she consumed 14.3 gallons. In miles per gallon, compute her rates of consumption both in city miles and in freeway miles.

SOLUTION. Lisa wants to know the values of "city miles per gallon" and "freeway miles per gallon." But, while the units of the rows in the table are miles, the information she has is that the sums across the rows in this table are in *gallons*. The equations she has to write down for each week are of the form:

$$(*) \qquad \text{gallons of city driving in the week} + \text{gallons of freeway driving in the week} = 14.3\ .$$

Remembering that

$$(**) \qquad \text{gallons} = \text{miles}\frac{\text{gallons}}{\text{miles}}\ ,$$

she realizes that can convert the data of the table into equations involving gallons by choosing as the variables "gallons per miles." Now the equation (*) becomes

$$(*) \qquad \text{city miles}\frac{\text{city gallons}}{\text{city miles}} + \text{freeway miles}\frac{\text{freeway gallons}}{\text{freeway miles}} = 14.3\ .$$

So, she labels the variables in which she is interested as C = city gallons per mile, and F = = freeway gallons per mile. Uusing these variables the rows lead to this system of equations:

$$131C + 210F = 14.3$$
$$180C + 120F = 14.3$$

Noticing that the numbers on the right side of each equation is the same, it is tempting to subtract the second from the first to get:

$$-49C + 90F = 0$$

or $F = (49)/(90)C$, which is already a startling fact: freeway driving is about 5/9 the cost (in terms of fuel used) of city driving. But to get the original problem, we start by replacing F in one of the original equations by this expression in terms of C, and then solve for C. We get $C = 14.3/245 = 0.058$. If we substitute this into either of the original equations, we can solve for F , to find that $F = 0.031$. Now, we get back to the original goal of the problem: to find the rate of fuel used with respect to miles for city driving and for freeway driving. These are the reciprocals of the values of C and F. Calculating those reciprocals, we find that, for Lisa's new automobile, city driving gets 17.13 city miles per gallon, and 32.25 miles per gallon on the freeway.

Chapter 5
Functions

In the preceding chapters we started to move our thinking about equations from *looking for a solution* to that of *expressing a relation* and of the use of letters to represent *unknown numbers* or *quantities* to that of *variables*. In both those cases we thought of x (or y or z or . . .) as a yet-to-be-determined number (or numbers) to be found by "solving" in the case of "unknowns", and "measuring" in the case of "quantities." But now we interpret the symbols x, y, z as *variables*; that is, they are to be understood as ranging over a whole set of numbers, and our interest in those variables is in understanding the relation expressed by the equation. As we shall see, this is not so hard if the relation is expressed as a graph, harder if expressed algebraically or by a table, and difficult if expressed by an algorithm. In all cases, we are moving from a *static* study of relations to a *dynamic one*: it is in this sense that letters represent "variables."

An equation with two variables x, y expresses a relationship between them. A solution of the equation consists of two specific numbers, one for each variable, which, when substituted in the equation makes a true statement. In case there is more than one solution, we may talk about *the solution set*. We usually use an ordered pair (x, y) to represent each solution. The order indicates which variable represent which number. Thus, the instruction "substitute $(5, -1)$ in the equation" means: set $x = 5$ and $y = -1$. For example, if the relation is $3x - 2y = 1$, then $(1, 1)$ is in the relation, but $(2, 3)$ is not.

In section 2 of chapter 3 we defined *relation* and *function* in rather abstract terms, and went on to illustrate by specific examples. In particular a *function* (written $y = f(x)$ and expressed as "y is f of x") is a set of instructions which produce, from a choice of specific number for x (called the input), a specific value for y (the output). Said another way, in any function, a given input does not give one output some of the time and a different output at other times. In this chapter, our focus will be on the relation between inputs and outputs, and not on the set of instructions that produce an output for a given input. To say this a different way, our interest is not on the details of the calculation of a y when given an x, but rather on questions like: if x gets larger, what happens to y? If x is halved, what happens to y? If x is replaced by $x + 2$, what happens to y? This is why we introduce the letter "f" to represent the set of instructions, without focusing on them. So, we can read "$y = f(x)$" as "start with x, do f to it, and record the output y."

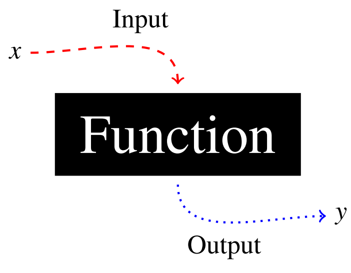


Figure 1

We now look at a function as a "black box" as in figure 1, so that we can concentrate on the relation between input and output, and move away from the mechanics of computing values of a function.

This chapter completes this transition from the concept of *unknown* to that of *variable*, and that from *equation* to that of *function*. We will focus on characteristics that separate linear from nonlinear functions. In the last section we discuss, in contexts, ways of expressing the relation among variables through various representations.

5.1 What is a Function?

1. Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.

2. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). 8.F.1,2

Example 1.

The following table is that of the bus schedule between Salt Lake City and Price.

Salt Lake City to Price										
LvSLC	8:00	9:00	10:30	12:00	13:00	14:30	16:00	17:00	18:30	20:00
ArrPrice	11:15	12:15	13:45	15:15	16:15	17:45	19:15	20:15	21:45	23:15

Examining the table, we notice several things: first of all, it takes the 8:00 bus three hours 15 minutes to make the trip; furthermore, this time is the time every trip takes. Also, the change between any two departure times is the same as the change between any two arrival times. Graphing these data (see Figure 2) makes these observations even more clear. The graph shows that the data lie on a line,

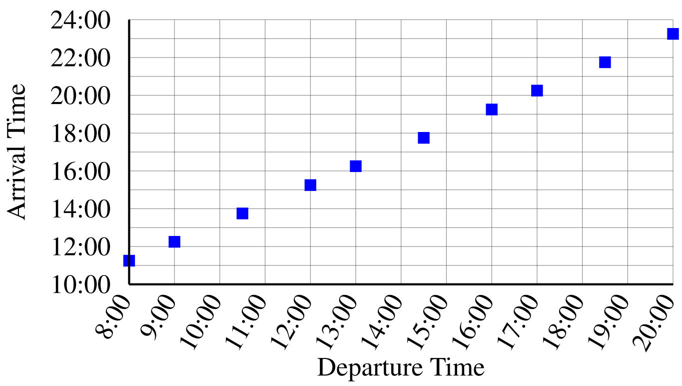


Figure 2. Bus Schedule

in fact, a line of slope 1, since any change in departure time results in precisely the same change in arrival time. We conclude that, whenever a bus leaves Salt Lake City, it arrives in Price 3 hours and 15 minutes later. This we call a *model* for the given data: in this case the model is linear. We use the model to show us immediately when a bus arrives in Price, for *any* given departure time from Salt Lake City.

If a new bus, with departure time 6:30 were to be added to the schedule, we schedule it to arrive in Price at 9:45. More generally, if a bus leaves SLC at D o'clock, it should be expected to arrive in Price at $D + 3 : 15$ o'clock. Letting A represent the arrival time, we arrive at this relationship between D and A: $A = D + 3 : 15$. We see that this formula tells us that the arrival time is completely determined by the departure time; that is A is a function of D. In such a statement, we consider A and D as variables in the sense that they can have any (time) value, and the relation $A = D + 3 : 15$ will hold.

Before going on, we note that, in the real world, arrival time is not completely determined by departure time; factors along the road may delay, or advance, the arrival of the bus. Figure 3: Real Data gives a more realistic graph of what may actually happen in a day.

This graph, of actual data, does give us important information: we should expect, on average, for the

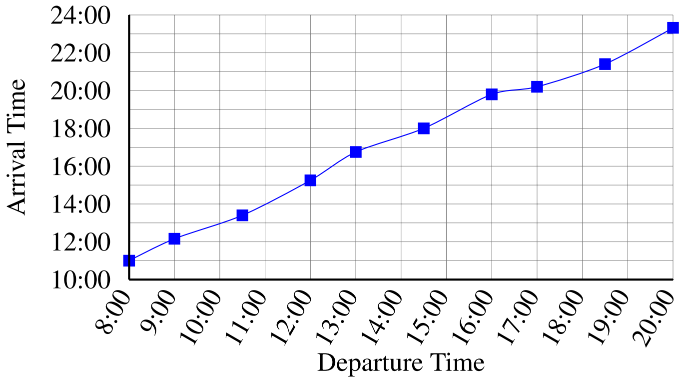


Figure 3

trip to take 3 hours and 15 minutes. However, in the early morning and late evening, the trip is likely to be quicker, while in the late afternoon, it is likely to take longer. We can still model these data with the straight line $A = D + 3:15$, but with the understanding that the arrival time is subject to traffic and weather conditions. We shall return to the subject of models for real data in Chapter 6. Our goal there will be to interpret tables of actual data so as to discover a curve, or a formula, that best models the actual data. For now, let us consider relations between two variables that give rise to functions.

Function Concept: Given two variables, x and y, we will say that y is a *function* of x if there is a set of instructions (which may be expressed as a formula, algorithm or recipe) that determine a specific y for a given x.

The notation used to assert that y is a function of x is $y = f(x)$, where f stands for the set of rules that tell us how to go from x to y. We read $y = f(x)$ as "y equals f of x. Of course, we may use other letters (such as g, h, etc. to represent other functions. This notation sometimes causes confusion, for students have become used now to using letters to represent numbers. So it is useful at this time to be clear that, for the first time, we are using a letter (f) to represent an *action*(implementing a set of rules) rather than a number.

Example 2.

$$y = 3x + 7$$

This can also be given by the set of instructions: pick a number x, multiply it by 3 and add 7. Notice that the instructions clarify the order of operations much better than the formula does, so it is good practice to translate formulas to sets of instructions to better understand them - in fact this is exactly what happens when we execute a sequence of operations on a calculator.

Example 3.

$$y = \frac{1}{x}, \quad x > 0 .$$

Here, we do not have a rule to give a value of y corresponding to $x = 0$. We say that the function is *not defined* for $x = 0$, or 0 is not in the *domain* of the function. For this function, x is a positive number, so we often make explicit that we are only interested in the function for positive values of x. For this function, we say that x and y are *inversely proportional* in the sense that if x is multiplied by any number, the y is divided by that number.

Plotting a set of values (x, y) that are related by a function provides a useful visualization of the function.The usefulness depends upon the extent to which the selected points illustrate the important features of the function. So, given rules describing a function, we create a set of points (x, y), where x is a number to which the rules apply (that is, x is in the domain of the function) and y is the number we get when applying the rule . When we plot enough points, we join them with a curve to get a representation of the function. For the general function this may take some skill or additional information contained in the context, but - as we have seen in the preceding chapter - for a linear function we need only find two points on the graph, and connect them with a line.

Let's go through this analysis for the above examples.

Example 2 revisited.

$$y = 3x + 7$$

Make a table of representative values

x	-2	-1	0	1	2	3	4
y	1	4	7	10	13	16	19

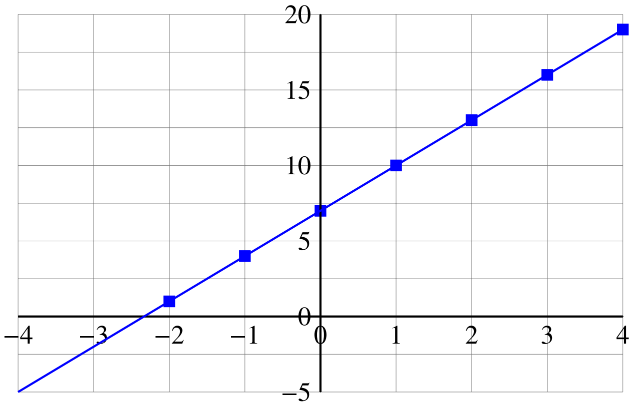


Figure 4

Example 3 revisited.

$$y = \frac{1}{x}, \quad x > 0 .$$

The rule here is "pick a positive number and take its multipleicative inverse." We create the table using the first eight positive half integers:

x	.5	1	1.5	2	2.5	3	3.5	4
y	2	1	.67	.5	.4	.33	.29	.25

Example 4.

The "Rambo Fliers" Come to Town. The local arena in Smalltown, Va., with occupancy limit 6000, hosted the fabulously popular drums and celestine group, the "Rambo Fliers" for a performance at 7 pm one Saturday night. Admission was at a fixed price, and doors opened at 4 pm with open seating. To accommodate those searching, the audience was admitted in groups (all those waiting to get in) every 15 minutes. The graph below is of the audience count at 15 minute intervals from 4 pm to 7 pm.

If we connect these lines, we get this:

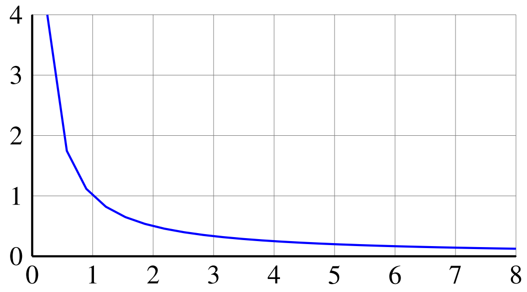


Figure 5

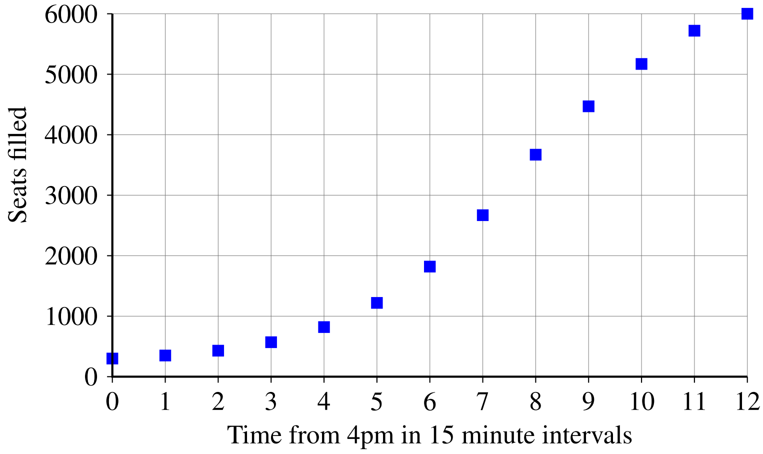


Figure 6. Occupancy at an Event

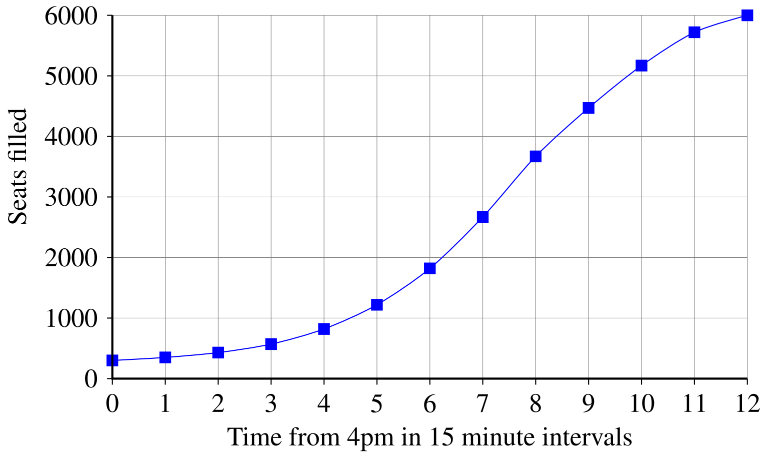


Figure 7. Occupancy at an Event

This is a very good graphic: it tells us a lot about likely arrival times at the concert, observations we might conjecture happen at any concert. At the beginning the audience flow is slow, but around 4:20 the rate picks up and stays strong until about 6:40, when people begin to settled own. So, the model created by the smooth connection is good for suggesting tendencies, but it is not an accurate portrayal of the actual event. Since people are let in at 15 minute intervals, the audience count remains constant during those intervals, and jumps to a new count at the 15 minute markers. So, the audience is a constant 300 from 4:00 to 4:15, and a constant 820 from 5:00 to 5:15 at which time it jumps by 400, and so on, The largest jump is a t 6:00, when 1000 are let in all at once. Thus, the accurate graph of population count looks something like this:

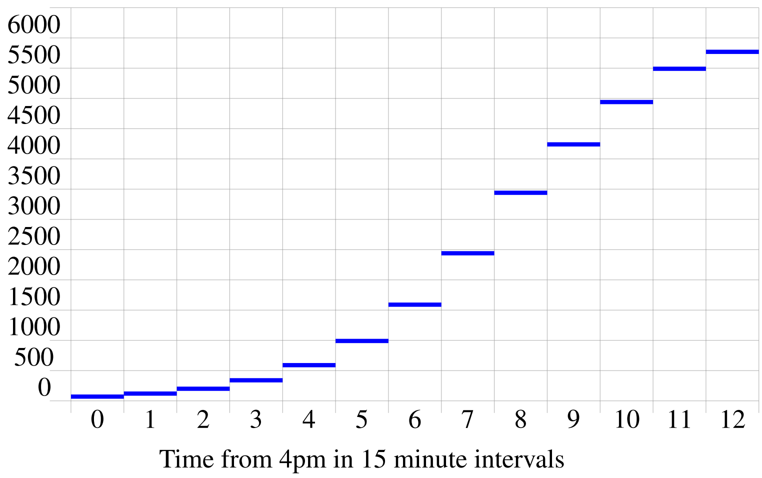


Figure 7. Occupancy at an Event

When we connect data points with a "smooth" curve we say that we are creating a *continuous* model of the process. In many cases, this is much more useful than the "actual" graph (which is *discontinuous*) For example if we kept an audience count of a football stadium that seats 100,000 people, the count is discontinuous (it takes time for one person to get through the turnstile, and furthermore the count is always a positive integer), but that level of detail is not of interest and in fact would be a distraction. Similarly, stock market purchases are not continuous, but they happen at such a large rate that anyone studying stock market behavior would use a continuous model (as we see in the business section of the newspaper).

Sometimes it is important to distinguish *discrete* processes (those that change at separated instances) from continuous processes (like water flowing down a river). For example, consider the use of electricity in a typical home. We use electricity at a constant rate between the times an appliance is turned on or off. At those instances, the rate changes abruptly. Since flicking a switch is an assault on the system (too many working appliances could throw a breaker). These instantaneous changes are significant features of the study of electricity use; especially when considering the rate of use for a whole city. Use of air conditioners on a very hot day could, and sometimes do, shut the whole system. City planners want to understand this phenomenon so that they can plan for its occurrence. Hence in this context it makes the most sense to look at the discontinuous graph joining the data points, while, in the context of filling a football stadium, the continuous curve tells us more.

Before proceeding, this is a good place to introduce a little terminology that goes along with the function concept. First, the set of values for which the function is defined is called the *domain* of the function. The *range* of the function is the set of values that can appear as the output of the function. Let's look over the preceding examples to identify the domain and the range of the function.

- Example 1. If a table of corresponding (x, y) values is given, then the set of x values is the domain, and the set of y values is the range. So, in example 1, the domain of the function is the set of values in the first row, and the range is the set of values in the second row. Figure 2 is a graph of that function. We went on to note (from either the table or the graph) that the difference between the y value and the x value is $3:15$ (using time notation). So, the equation $y - x = 3:15$ holds for all corresponding values of the function. Then we went on to model the given data by that equation, which defines a new function, whose graph is the straight line through the set of values of the first function. This new function is defined by the rule $y = x + 3:15$, whose domain and range are both the set of all possible times of the day. This gave us a way of predicting arrival times for buses leaving Salt Lake City at any time.

- Example 2. Here the function is defined by the rule: $y = 3x + 7$. To graph the function, we pick a set of values of x, and calculate the corresponding value of y, thus creating a table. Then we plot the points on a grid, and join these points with a line. It is important to stress that in this case the line *is* the graph of the function; we created a table by picking values o fx arbitrarily, and then following the rules of the function. In fact, since we recognize the defining equation as that of a line, we need only have picked two values for x. Since the domain of a functions is the set of values to which we can apply the rules, the domain of this function consists of all numbers on the number line.

- Example 3. In this example, we have already specified the domain: $x > 0$. Since the multi plicate inverse of a positive number is positive, the range also is the set of positive numbers. Now, we might ask, what about negative numbers? Yes, the equation $y = 1/x$ does allow for a negative x, and for such an x the output will be negative. So, potentially, the equation $y = 1/x$ defines a function whose domain (and range) consists of all nonzero numbers. However, the proposer of the function is the one who gets to decide on the domain. Should she say that we want to only consider odd whole numbers divisible by 17, then that is the domain. But, in this case the proposer specified that the domain consists of all numbers greater than zero, and we conclude that this is also the range is the set

- Example 4. This is interesting because each time we make an improvement on the graph, we change the domain and range of the function; To begin with, the table of values is the set of all times at 15 minute intervals between 4:00 and 7:00, with the first data point recorded as 0 and the last as 12.

x	0	1	2	3	4	5	6	7	8	9	10	11	12
y	300	350	430	570	820	1220	1820	2670	3670	4470	5170	5720	6000

The range is the set of positive integers on the second row of the table. These points are graphed in Figure 6. But now, in Figure 7, we've modeled the data with a smooth curve. Here, the domain is the set of *all* times between 4:00 and 7:00, and the the range, the set of all numbers between 300 and 6000. Our last observation was that it doesn't make sense to say that there are 1035.76 people in the stadium at 5:08; In fact there are 820 and that has been true since 5:00 and will be true until 5:15. So, we drew the more accurate graph, Figure 8, of a function whose domain is all times between 4:00 and 7:00, but whose range has returned the set of positive integers in the second row of the table.

Functions Defined by Graphs

A graph can represent a function using this as the rule:

Given a value for x, draw the vertical line through that value on the x-axis. Where it hits the graph, draw the horizontal to the y-axis, That point is the value of y corresponding to the given value of x.

For this rule to work, we must know two things:

a. for a given number a, the vertical line $x = a$ intersects the graph;

b. for a given number a, the vertical line $x = a$ intersects the graph only once.

If these two conditions are satisfied, then the rule works: the vertical line through a (on the x-axis) intersects the graph at one point, and the horizontal line through that point intersects the y axis at some point b. This b is the value of the function for the input a.

If either condition fails for a number a, then the function cannot be defined at a. We express this by saying that a is not in the *domain* of the function. Just to say this another way, a graph defines a function for all numbers a for which conditions **a** and **b** hold. That set of numbers is the *domain* of the function, and for any a in the domain, the above rule produces the value of the function at a. In some cases, if condition **b** fails: that is, the vertical line through some points on the x axis intersects the graph in more than one point, it may be possible to add a rule to the definition of the function that picks the correct point on the graph. For example, consider the graph of $y^2 = x$. For negative values of x, there is no intersection point of the graph with the vertical line, so no negative number is in the domain of the function. $x = 0$ produces only $y = 0$, so 0 is in the domain of the function. However, if $x > 0$, the vertical line intersects the graph in the positive and negative square roots of x. If we add the rule "y is not negative," then we have describe a function for all non-negative numbers x: y is the non-negative square root of x. This function is denoted $y = \sqrt{x}$. We will see this again in example 9.

Example 5.

Create a data table for points on the graph in figure 9.

Applying the rule, we can create this table of values of the function:

x	-3	-2	-1	0	1	2	3
y	11	8	5	2	-1	-4	-7

First we observe that the graph is a straight line. We can pick any two points to find the slope of the line. Let's choose $(-2, 8)$ and $(1, -1)$ and calculate the slope:

$$m = \frac{8 - (-1)}{-2 - 1} = \frac{9}{-3} = -3$$

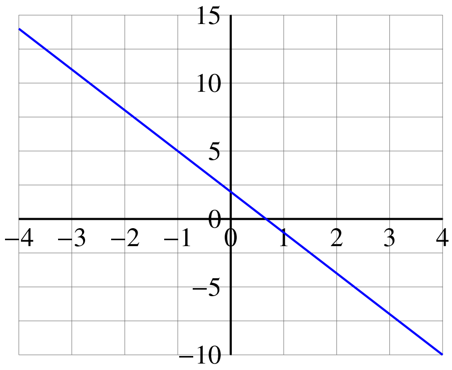


Figure 9

Since the y-intercept is given by the point $(0, 2)$, we know that $b = -2$, so the equation for this graph is $y = -3x + 2$.

Example 6.

Create a data table for points on the graph in figure 10.

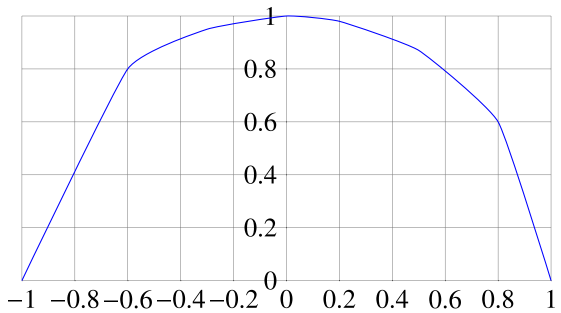


Figure 10

x	-1	-.6	-.3	0	.2	.5	.8
y	0	.8	.95	1	.98	.87	.6

In example 5, the graph went through integer points so the pairs (x, y) were easy to find according to the set of rules for graphs. In this example we have to pick values of x for which we could most easily estimate values of y. In any case, this is not a line.

Example 7.

Federico and Nkutete are hired at the same time by the Boston Envelope Company. However, they have different compensation contracts. Federico will start at an annual salary of $25,000, with guaranteed raises of 6% each year, while Nkutete starts at an annual salary of $20,000, with guaranteed annual increments of $1,000. Which contract is better?

Solution. The answer will depend upon how long they intend to work there. So, for example, at the end of the first year, each gets a $1,000 raise, so, Federico still earns more than Nkutete. The next year, Federico will get a slightly higher raise, but still has a lower salary. In fact, let's tabulate the effect on the salary of the annual raises for the first 12 years.

Year	0	1	2	3	4	5	6	7	8	9	10	11	12
Federico	20000	21000	22050	23153	24310	25526	26802	28142	29549	31027	32578	34207	35917
Nkutete	23000	24000	25000	26000	27000	28000	29000	30000	31000	32000	33000	34000	35000

By the twelfth year, Federico will have a higher annual salary, but, his cumulative income of \$318,343 is about \$24,000 less than Nkutete's cumulative income of \$342,000. Besides, it is altogether likely that they will both get promoted within the first ten years, so, the reasonable response to the question is that Nkutete is getting the better deal. Let us look at the graph of the data (Figure 11)

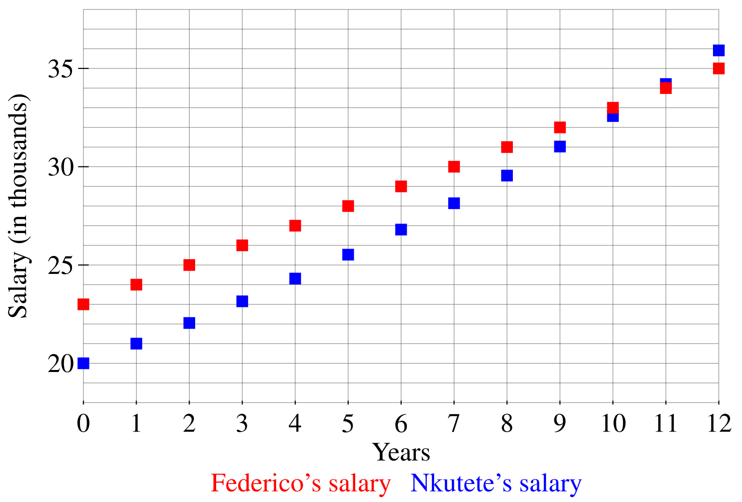


Figure 11

The graph clearly shows that, although Federico starts out with a higher salary, the gap between the salary decreases over the year, and ultimately, Nkutete is the higher earner. But what if they both worked at this same job for 40 years? In that span, who has gotten the better deal and by how much? We will return to this example in the next section, where we will see some differences between these two graphs that can shed light on this question.

5.2. Linear and Nonlinear Functions

Interpret the equation $y = mx + b$ as a linear function. Observe that if b is not zero, the variables are not in proportion; however, the change is in the variables between two points are in proportion (hence the idea of slope). 8.F.3

Distinguish between linear and nonlinear functions.

The characteristic of a proportional relationship is that the quotient y/x, for values in the proportion, is always the same, and we call this the unit rate of y with respect to x. The significant characteristic of a line is this: for any two points P and Q, the ratio of the change in y from P to Q to the change in x from P to Q is a constant, called the rate of change of y with respect to x. This is an important characteristic: the variables in a linear relation are in proportion *only* when the graph of the relation goes through the point $(0,0)$. To see this algebraically: if y is a linear function of x; that is $y = mx + b$, the the quotient y/x, for $x \neq 0$ is

$$\frac{y}{x} = m + \frac{b}{x}$$

which is definitely not constant for $b \neq 0$.

Important things to remember about linear functions are:

If the line intersects the y-axis in the point (0, b), then the equation of the line is y = mx + b.

If the line is horizontal, the slope is zero, and the equation of the line is $y = b$.

If the line is vertical, it has no slope, and its equation is $x = a$.

If the line goes through the origin, the equation of the line is $y = mx$ and the values of y are proportional to the values of x; otherwise said, $y/x = m$.

If the line has slope m, and the point (x_0, y_0) is on the line, then the equation of the line is

$$y - y_0 = m(x - x_0)$$

If (x_0, y_0), (x_1, y_1), then a point (x, y) is on the line if

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0}$$

At this point, it is desirable to look at a variety of examples of functions in various representations (formula, table, graph) to make clear the distinction between linear and nonlinear functions. To illustrate this need, let's go back to the hires of the Boston Envelope Company. The data in Figure 11 shows the gap in salaries decreasing, but it doesn't make clear what the relationship will be in the long run. For that it is desirable to model these data by connecting the points with a curve that doesn't introduce any extraneous information (see figure 12).

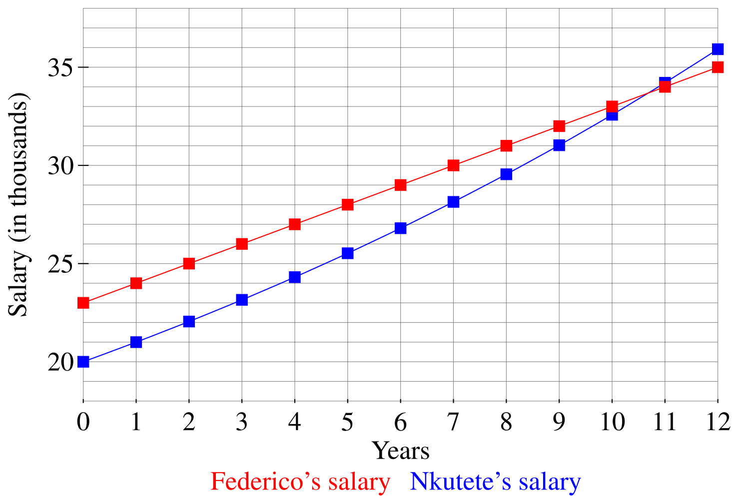


Figure 12

The curve modeling Federico's salary is a straight line, while that for Nkutete is not; neither the table nor the graph of data points showed a tendency for Nkutete's salary curve to become steeper and steeper. If we calculate with the table, we can see that the rate of change of Nkutete's salary gets larger over time; the value of the graph is that it shows us this instantly.

As we continue to study data for two variables, looking for a relation between them, we hope to find a formula that actually exhibits one variable as a function of the other. This will allow for prediction of future pairs of values not on our table. To set the ground for this, we look at a collection of examples represented in various ways: formula, tables or graphs.

Example 8.

$y - 2x = 11$

When we start, as in this case, with a formula relating two variables, it is not clear which is a function of the other, if at all. Often the context tells us what choice to make, other times it is desirable for reasons of computation, to make a choice. In this case, we could write y as a function of x:

$$y = 2x + 11$$

or x as a function of y:

$$x = \frac{y - 11}{2}$$

Since the first is simpler, let's make that choice, and create a table like this:

x	-4	-3	-2	-1	0	1	2	3	4
y	3	5	7	9	11	13	15	17	19

Notice that every time x increases by 1, y increases by 2. Recall that this tells us that 2 is the slope of the line, or the unit rate of change. See the graph (Figure 13, next page).

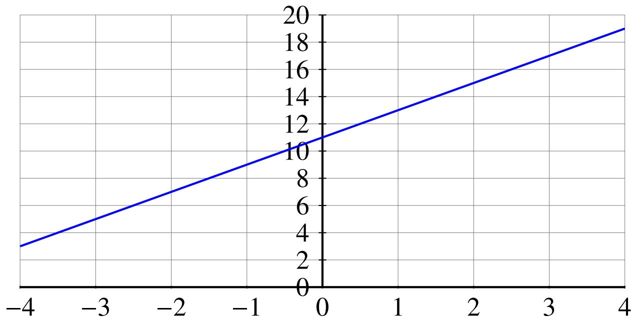


Figure 13

Example 9.

$y^2 = -x + 1$

Here, it is easier to make the table by writing the relation in the form $x = 1 - y^2$ and finding values of x corresponding to values of y. This gives us the table and graph

x	-8	-3	0	1	0	-3	-8
y	-3	-2	-1	0	1	2	3

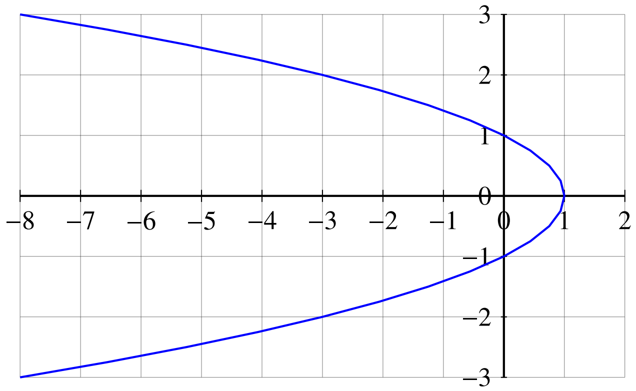


Figure 14a

We see from this graph that this does not specify y as a function of x; at least not until we include a rule that tells us, for any x which of the two candidate values is to be chosen. We may, for example, add the rule: For each x, let y be the positive number such that $y^2 = -x + 1$. Then we get this graph, which now describes a function:

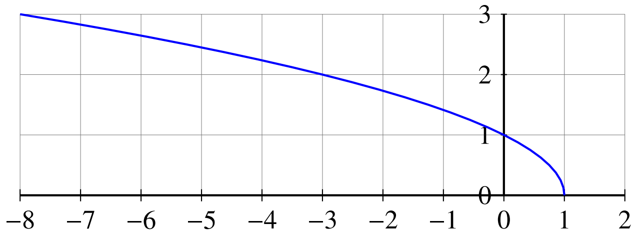


Figure 14b

To summarize: figure 14a (of the relation $y^2 = -x + 1$) does not describe y as a function of x, because of the ambiguity in taking the positive or the negative solution of $y^2 = -x + 1$. This ambiguity is resolved by adding the stipulation: For a given $x < 1$, let y be the positive solution of the equation $y^2 = -x + 1$, resulting in figure 14b.

5.3. Modeling and Analyzing a Functional Relationship

This section - like many of the topics in 8th grade - is exploratory, with the goal of understanding functional relationships in context. There are two processes to be introduced and explored. First, suppose that we are considering two variables (for example: the height and girth of a maple tree) that we think might be functionally related. We gather data on the variables, specifically pairs of the values of the two variables in a sample set, or in experiments. Finally, we study the variables in a variety of ways to see if we can find a model (a specific set of rules defining a function) that fits the data well enough to be able to make predictions on the outcomes of further sampling or experimentation. Second, we may be given a functional relationship, that is, a set of rules that determine values of the second variable dependent upon values of the first. This may take the form of an equation involving the two variables, or an algorithm to compute one from the other. In this case we study the functional relationship in a variety of representations (tables, graphs and equations) to see if we can understand the properties of the relationship. In this chapter we shall look at what mathematicians call the *deterministic* data; for example two different ways of measuring the same physical attribute. In the next chapter, we explore how to (best) do this with data gathered at random, and thus subject to random inputs. In our first few examples, the context clearly indicates a constant rate of change, and thus, a linear relationship. The subsequent examples show a variable rate of change; here we explore what we can learn from the data. In every case, we make a choice of one of the variables, x as the variable upon which the other variable, y is dependent. We determine, from the graph, in what intervals y is increasing/decreasing as x increases, and we begin to understand the significance of those points where this behavior changes.

Constructing Functions

Construct a function to model a linear relationship. Determine rate of change, initial value (use representations + context). 8.F.4

Example 10.

Heat and Temperature

The temperature of an object measures the amount of heat it contains. Temperature is measured in degrees, denoted °. No matter what scale is used, it should be such that a change in the temperature of an object is proportional to the change in heat content expressed in some other measure, such as calories. However, caloric content is hard to measure directly, and so we turn to other means to quantify heat. For example, some fluids expand in volume as they heat up, and in a linear way: the change in volume is proportional to the change in caloric content. Mercury is such a fluid, and thus is the fluid of choice in a thermometer. As the object heats up, the mercury expands and the column of fluid in the stem of the thermometer rises. The important thing is that this change in height of the column of mercury is proportional to the change in volume, and thus proportional to the change in heat.

The Celsius temperature scale is based on water: 0 °C corresponds to the heat content of newly melted ice, and 100°C to water just starting to boil. Thus, if a pot of water measures 50°C,the increase in caloric content from 0°C is half the increase in caloric content of the same amount of water at the boiling point. Daniel Gabriel Fahrenheit was a doctor in the 18th century who wanted to measure the heat generated by a disease in a human patient, so he invented a scale that was based on humans: 100 °F is the temperature of a healthy human being, and 0°F is the temperature of blood just about to freeze. So, for example, if a person shows a temperature of 102 °F, that person is 2% hotter than a healthy person. By experimentation, Fahrenheit discovered that, in his temperature scale, the freezing point of pure water is 32°F, and the boiling point of water is 212 °F.

Given that these two temperature scales are linear with respect to caloric content, they are linear with respect to each other. So we can relate °C with °F by a linear relation. We know two points on the graph of this relation: the freezing point of water, $(0, 32)$ and the boiling point of water, $(100, 212)$ (where we have put °C as the first coordinate). The slope of the line graphing this relation is

$$\frac{212 - 32}{100 - 0} = \frac{9}{5}$$

This can be stated this way: a 9 degree change Fahrenheit is the same as a 5 degree change Celsius. Now we also know the y-intercept: it is 32, since (0,32) is on the graph. Thus the function relating Fahrenheit to Celsius is

$$F = \frac{9}{5}C + 32$$

Now, we can express Celsius as a function of Fahrenheit, by solving for C in terms of F:

$$C = \frac{5}{9}(F - 32)$$

Example 11.

At Mario's Cut Rate Used Car Lot, Mario compensates his salespeople with salary + commission: each salesperson receives a base salary and then a certain amount for each car sold. His more experienced people get a higher base salary, but the new people get a higher commission, because he want to encourage them to be eager to sell cars. Sally, his most seasoned salesperson receives a salary of $4,000 per month and a commission of $250 per car sold. Dmitri is a rookie, receives a salary of $2,800 per month, but his commission is $325 per car sold. Now, let's see what we can learn from examining these two means of compensation.

Here a table of Sally's and Dmitri's earnings at 0, 1,2,3,. . . ,8 cars sold.

Cars Sold	0	1	2	3	4	5	6	7	8
Sally	40	42.5	45	47.5	50	52.5	55.0	57.5	60
Dmitri	28	31.25	34.5	37.75	41	44.25	47.5	50.75	54

Now we graph the data, and put lines through the points corresponding to Sally's compensation, and those points for Dmitri. We know that they all lie on a line, because of the constant rate of change ($250 per car for Sally and $325 per car for Dmitri).

As these lines have different slopes, they intersect. The point of intersection tells us the number of cars each must sell in order to have the same salary. From the graph, we can read off the coordinates of that point, or we can do that algebraically: Let N represent a number of cars sold, and S, Sally's compensation for N cars sold, and D, Dmitri's compensation for N cars sold. The rules can be written algebraically as

$$S = 4000 + 250N \quad , \quad D = 2800 + 325N$$

At the point of intersection $S = D$, so we have to find out, for what N is $4000 + 250N = 2800 + 325N$? The solution of that equation is $N = 16$: at 16 cars sold, Sally and Dmitri receive the same compensation, $8,000.

Example 12.

A bookseller is trying to set a price for her books in such a way as to keep the carry-over inventory at an acceptable level. So, she decides to vary her prices, month by month for a little over a year, to see the relationship between price and inventory. Here are the data:

Month	Oct	Nov	Dec	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep
Price	1.4	1.10	1.00	1.4	1.80	2.40	1.60	2.00	2.50	3.50	2.65	1.50
Inventory	90	98	75	55	98	146	115	100	110	175	125	92

We can't tell much from these data, except that each time the price was lowered, the unsold inventory was lowered. Maybe, if we reorder according to price, with the inventory as the dependent variable, we get a different picture. In fact the picture we get is this:

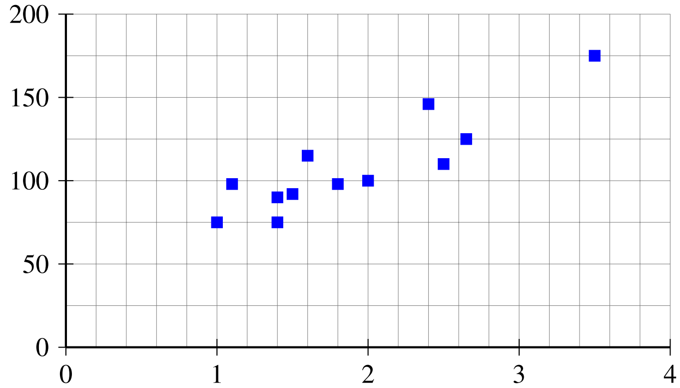


Figure 15

We see several things that are not readily apparent from the table: generally speaking, as the price rises, so does the inventory. We could conclude that, if we never want an unsold inventory of more than 150 items, then we should keep the price under $2 (well, almost all the time, in one month out of 10 this was not true). We also don't discern any curving of the data, so we might surmise that the relation (except for random variations) is linear.

In general, if we are given a table of data, we should first determine (from the context) which variable should be the horizontal, and which, the vertical. Then we should reorder the table in increasing order in x. We can now check for linearity: if the change in y is proportional to the change in x (that is, given any two points, the quotient of these changes is always the same number), then the data are that of a linear relation. An easier way, and one which in any case gives good information, is to plot the points to see whether or not they lie on a line. The data may have come from measurements which are prone to random error. So, if the points almost lie on a line, but do not actually lie on a line, we may be able to conclude that the relation is linear.

Analyzing a Functional Relationship

Data that are collected from real contexts, such as a laboratory experiment, or a questionnaire, are very unlikely to fall on a line - or for that matter in any precise pattern. Nevertheless, the graphed data may show a trend or suggest a relation, or uncover an anomaly.

Example 13.

On average over the past 175 years, the hour by hour temperature, from 1 am to 9 pm, for an August day in Salt Lake City is:

Time of day:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
Temperature:	73	72	71	70	69	69	70	72	76	81	74	84	85	87	89	90	89	88	68	85	84

Figure 13 is a plot of the points on a graph. We have connected those points with a smooth curve:

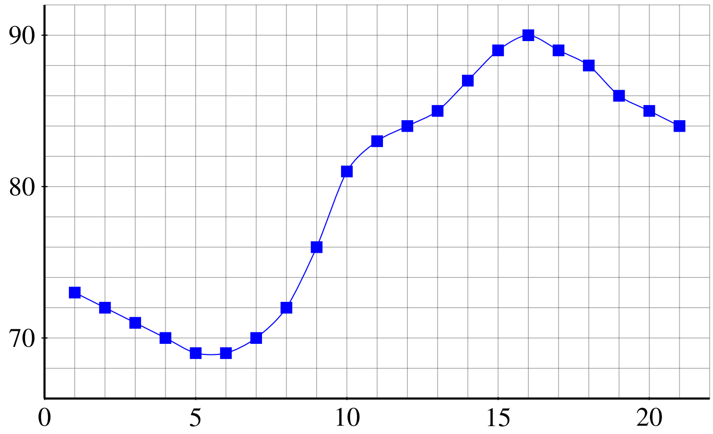


Figure 16: Average August Temperature

The graph confirms some things that we should have expected on physical grounds: that the temperature rises during the day as the sun moves directly overhead, and drops - more or less linearly - when the sun is down. We also see that the highest temperature is later in the day than we might have suspected; suggesting a cooling-off lag.

Now let's look at the data for a particular day: August 26, 2012:

Time of day:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
Temperature:	78	76	75	74	73	72	73	75	77	82	86	90	92	94	84	78	82	84	82	81	80

Here is the graph:

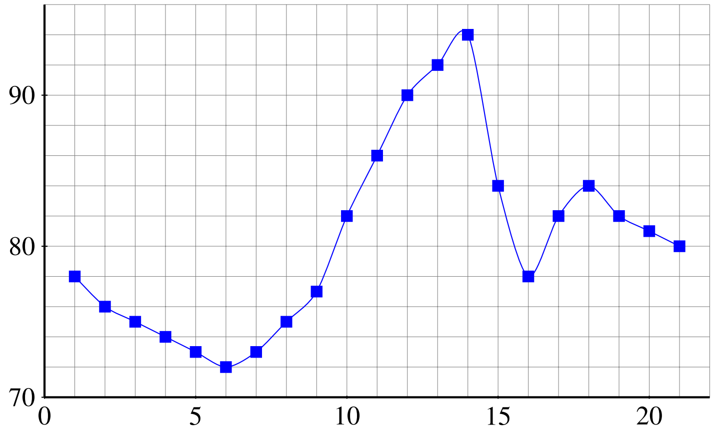


Figure 17:Temperature, Aug. 26,2012

This starts out like a typical August day, but then there is a sudden decrease in temperature at 15:00 for about 2 hours. This suggests a thunderstorm or the arrival of a cold front. However, the data show a warming up around 17:00, returning to the "typical day." This is not what happens when a cold front arrives, so strengthens the argument for a thunderstorm.

Let's look back at Figure 16, of average temperatures for August. Notice that the temperature rises at a steady state until about 3 pm where it flattens out a bit. Since this is the average temperature, this blip suggests that afternoon thunderstorms occur in August frequently enough to affect the average. This example is important because it illustrates the difference between "average" and typical. Figure 16 is not the temperature picture for a typical day, for there are (at least) two typical days, one with no afternoon thunderstorm (in which case the temperate will rse steadily until about 5 pm, and the other typical day with a thunderstorm.

Eventually, we will discover that there are just a few most important graphs that come up while trying to model situations. The following set of problems ask this: given a graph, can one describe a process, or situation that is modeled by this graph?

Example 14.

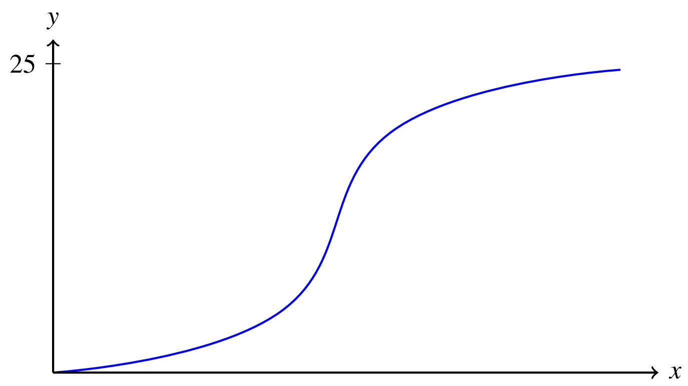


Figure 18

The curve in Figure 18 starts at $(0, 0)$. As x increases, at first y increases slowly, then during the next period increases rather rapidly, and finally levels off at about $y = 25$. If you have ever gone to a concert in a basketball arena, you have seen this behavior. In this model, time is on the x axis, and the y axis shows the number of people in the arena (in thousands). At first, people dribble in slowly, and the rate at which people enter the stadium rises rapidly, then rapidly lowers. Just as the game starts, the remaining seats get filled more and more slowly.

Example 15.

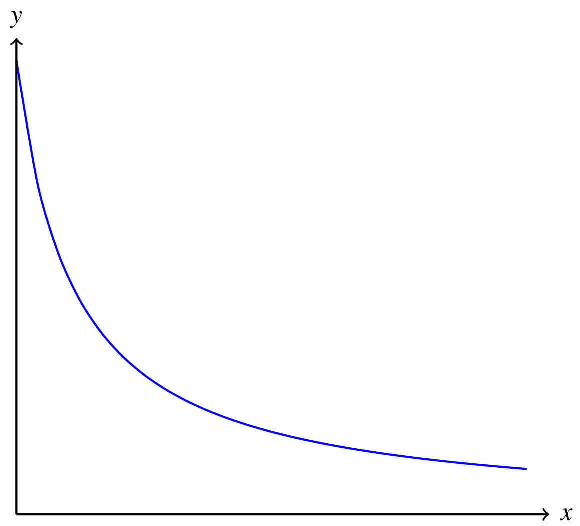


Figure 19

Figure 19 looks like the profile of a ski run, suggesting that it is a graph of altitude against time as I ski down a black diamond run that levels out at the bottom.

Example 16.

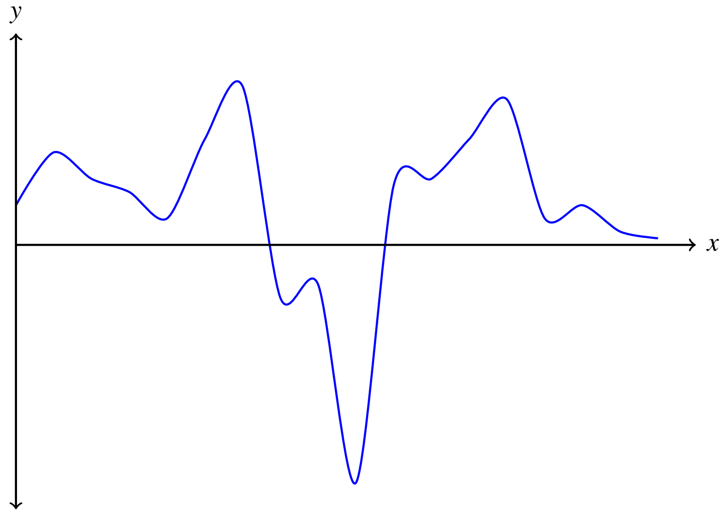


Figure 20

Now this looks like a random graph, but our task is to find a context which might lead to this graph. One thing that comes to mind, is that these are two islands separated by a deep trench. The x-axis can be interpreted as sea level. In particular, x is distance along a straight line cutting through both islands, and y is the altitude above (or below) sea level.

Chapter 6
Investigate Patterns of Association in Bivariate Data

In 8th grade, students investigate patterns of association in numerical (or "quantitative") bivariate data by constructing and interpreting scatter plots. An emphasis is placed on informal linear association analyses. In addition to using linear models to solve problems regarding numerical data, students explore categorical (or "qualitative") bivariate data through construction and interpretation of two-way frequency tables.

It should be noted up front that the practice of statistics is fundamentally different from the practice of mathematics. Thus, the integration of statistics within a mathematics curriculum is potentially misleading, with respect to the natures of both mathematics and statistics. While mathematics draws logical deductions from a set of axioms, statistics does not. Instead, statistics endeavors to quantitatively communicate properties of and relationships among observable phenomena. To that end, statistics is more like a scientific discipline than mathematical (e.g., the use of statistics is evidence-based, as its subject matter is analysis of data; statistical conclusions change over time, as more data are collected in various ways; mathematical conclusions do not change, as propositions are put to rest after being proved or disproved). To correctly convey the powers and weaknesses of statistics, teachers need to be steadfastly aware of the language they use when communicating statistical ideas to students.

Although some of the following italicized words are not mentioned in the Utah core for the 8th grade, it may be a good idea to casually demonstrate their usage while communicating with students.

An *experiment* is an activity for which *outcomes* occur randomly (i.e., based upon chance). The *sample space* of an experiment is the set of all possible outcomes. A *random variable* is a function that maps the sample space of an experiment to the set of real numbers. *Realizations* of a random variable are the specific values that the random variable may assume. A random variable can be thought of as a quantity that can assume more than one value, based upon chance events. We discuss two kinds of random variables: *quantitative* random variables and *categorical* random variables. Quantitative random variables are those of cardinal numerical value (e.g., 3 feet; 2.73 gallons; 4 children), whereas categorical random variables are those representing some quality or name. This distinction must be made clear in practice, since a set of categories can be easily replaced by nominal real numbers.

Now that we have introduced some general statistical concepts, we turn our focus to the 8th grade, which concerns itself specifically with bivariate data. A *bivariate data set* is a set of ordered pairs (x, y), where x and y are realizations of two different random variables (X and Y), such that the specific realizations x and y correspond to each other in some way (e.g., the ordered pair describe the same individual, they describe the same time period, or they may be related through some other such rule of correspondence). The nature of the correspondence between specific realizations x and y is described in the following examples.

Example 1.

Let X be the random variable "the European shoe size of a citizen of Springville", and let Y be the random variable "the height in centimeters of a citizen of Springville". Certainly, X and Y may assume a myriad of realizations. Maria conducts an experiment by recording the shoe sizes and heights of 53 randomly-sampled Springville citizens (this is the sample space). The bivariate data collected by Maria is the set of all 53 (x, y) such that x and y correspond to the same citizen. The "relation" between each specific realization x and y in the bivariate data set is that each pair describes the same person's shoe size and height.

Example 2.

Let X be the random variable "the average cost of gasoline in Cedar City in a given year", and let Y be the random variable "the number of speeding tickets written in a given year in Iron county". Lucas conducts an experiment by looking up and recording the realizations of X and Y from the year 1972 through 2012. Lucas decides to pair each x-value with the y-value that corresponds to the same year. Thus, Lucas's bivariate dataset is the set of (x, y) pairs for each year in the given 41 year range.

Section 6.1: Construct and Interpret Scatter Plots for Bivariate Data

Construct and interpret scatter plots for bivariate measurement data to investigate patterns of association between two quantities. Describe patterns such as clustering, outliers, positive or negative association, linear association, and nonlinear association. 8.SP.1

Here, when the Utah Core refers to "quantities", it means "quantitative random variables". People are often interested in whether or not one random variable is associated with another. For example, is there a relationship between the number of television commercial broadcasts of a certain product and the number of sales of that product? Do 8th grade students who can do many push-ups in P.E. class also tend to be able to do more pull-ups? This section discusses methods of answering such questions about quantitative random variables such as "number of push-ups".

Conventionally, X is assigned to be the *input* random variable (i.e., the *independent'* or "explanatory" variable) from which we wish to predict the *output'* variable Y (i.e., the (presumed)*dependent'* or "response" variable). Of course, one does not initially know if there exists any dependence structure between the two; quantifying the relationship between X and Y, based upon observed data collected through random sampling, is precisely the goal of our statistical analysis here. It is important to note that there is an obvious relationship between the two specific realizations making up a single datum (i.e., a specific (x, y) pair), namely that there is some rule of correspondence, such as the fact that they describe aspects (or qualities) of the same individual or time period. However, the investigator is not interested in the relationship between realizations making up a single datum. The investigator wishes to quantify the relationship between two random variables, X and Y , which describe an entire population, not between the specific values x and y for a given subject. This point is subtle, but extremely important, lying at the very heart of statistics: One cannot make inference on a population based upon an individual, nor vice versa.

To visually inspect the potential influence of one random variable on another, we depict our sample data on a scatter plot. A *scatter plot* is a graph in the coordinate plane of the set of all (x, y) ordered pairs of bivariate data. Consistent with the usual convention, we place the independent variable X on the horizontal axis and the dependent variable Y on the vertical axis.

Example 3.

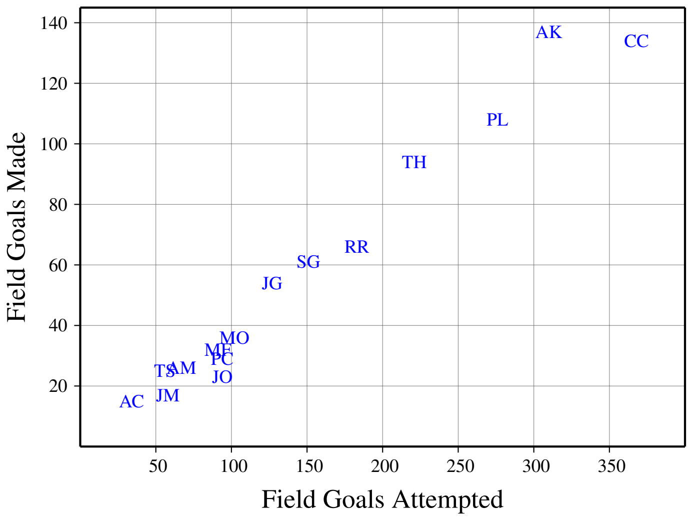
Izumi is the score keeper for her school's basketball team. Izumi's responsibilities include keeping a record of each player's total number of field goals made, the total field goals attempted during the season, the total number of assists and the total number of rebounds. For those not familiar with basketball, let us define these terms. Basketball is a game involving 5 players on each of two sides, using a ball and "goals." The playing field is a rectangle with "goals" at each short end of the rectangle. The goal is a basket set 10 feet off the floor of the playing field, and the object of the game is to put the ball in the basket. Any shot at the basket is an "attempt," and if the ball goes through the basket, this is a "goal."

Goals can contribute two or three points, depending upon the distance covered. For Izumi's purpose this is not important: the important data are "goals attempted" and "goals made." An "assist" is awarded to a player who delivers the ball to someone who actually makes a goal. Finally, a "rebound" is awarded to a player who catches the ball when a goal is attempted, but not made. While the names of players have been changed, these data (for the 2012-13 season) were borrowed from actual Utah high school girls basketball players via www.maxpreps.com.

Part of Izumi's duties include helping the coach decide which players deserve awards at the end of the season. Izumi notes that Ameila Krebs was the highest-scoring player for the season, but Amelia also had a high number of failed field goal attempts. Izumi would like to further investigate the relationship between the two random variables "Field Goals Made" and "Field Goals Attempted". Izumi's data are given in the table below.

Player	Field Goals Attempted	Field Goals Made
Amber Carlson	34	15
Casey Corbin	368	134
Joan O'Connell	94	23
Monique Ortiz	102	36
Maria Ferney	91	32
Amelia Krebs	310	137
Tonya Smith	56	25
Juanita Martinez	58	17
Sara Garcia	151	61
Alicia Mortenson	67	26
Parker Chistiansen	94	29
Rachel Reagan	183	66
Paula Lyons	276	108
Thao Ho	221	94
Jessica Geffen	127	54

To better visually inspect her data, Izumi makes the following scatter plot of Table 1, where each data "point" consists of the initials of the corresponding basketball player.



2012-2013 Girls Basketball Data

Because Izumi would like to visualize each individual player, she chooses to identify them by their initials. However, if she were more interested in the relationship between her two variables (Field Goals Made and Field Goals Attempted), then she would likely make a plot with a marker for each data point (see the plot on the next page).

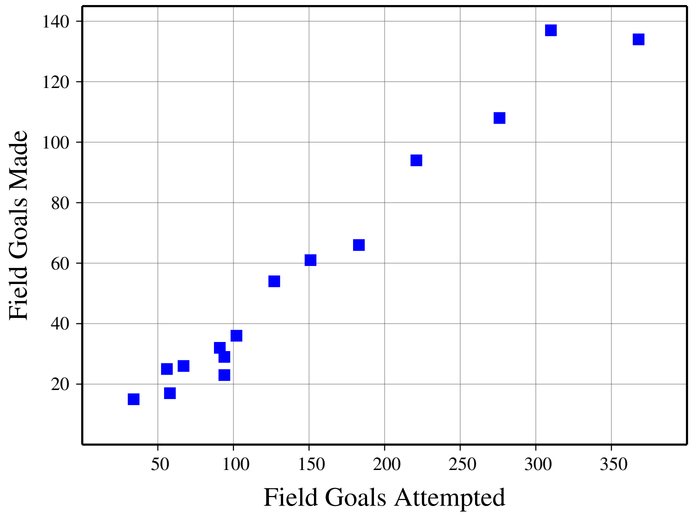


figure 1: 2012-2013 Girls Basketball Data

It should be noted that it is the latter plot with the points (rather than the initials) that is typically created. We include the plot with the individual subjects identified by initial for two purposes:

1. To provide an intermediate step between the data table and the typical scatter plot;

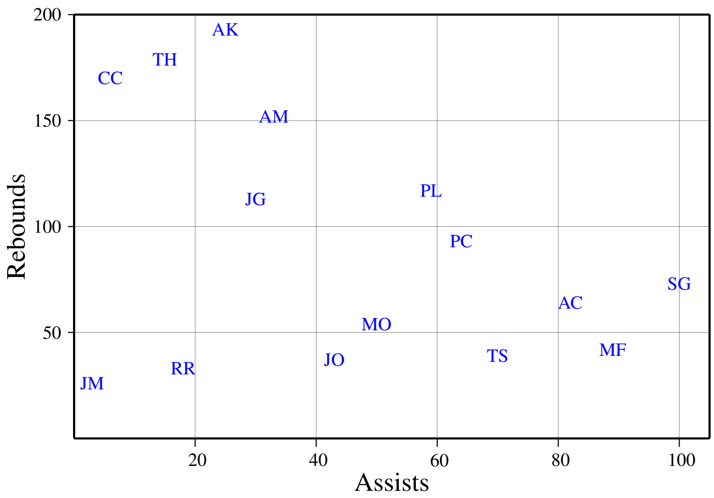
2. To be able to reference individual players during the data analysis discussion. The nature of the association between the variables Field Goals Made and Field Goals Attempted will be discussed in the next section.

In addition to data about field goals, Izumi is curious about the relationship between the number of assists a player makes and the number of rebounds a player makes in a season. She notices that the players who make the most assists tend be in positions located far away from the basket. Izumi's Assist and Rebound data are given in the table below the plot on the next page.

Player	Assists	Rebounds
Amber Carlson	82	64
Casey Corbin	6	170
Joan O'Connell	43	37
Monique Ortiz	50	54
Maria Ferney	89	42
Amelia Krebs	25	193
Tonya Smith	70	39
Juanita Martinez	3	26
Sara Garcia	100	73
Alicia Mortenson	33	152
Parker Chistiansen	64	93
Rachel Reagan	45	67
Paula Lyons	59	117
Thao Ho	15	179
Jessica Geffen	30	113

From these data, Izumi creates the scatter plot below, again using the players' initials to identify each individual.

Again, note that if Izumi were interested less in individual players and more in the general relationship between assists and rebounds, she would have made the next scatter plot.



2012-2013 Girls Basketball Data

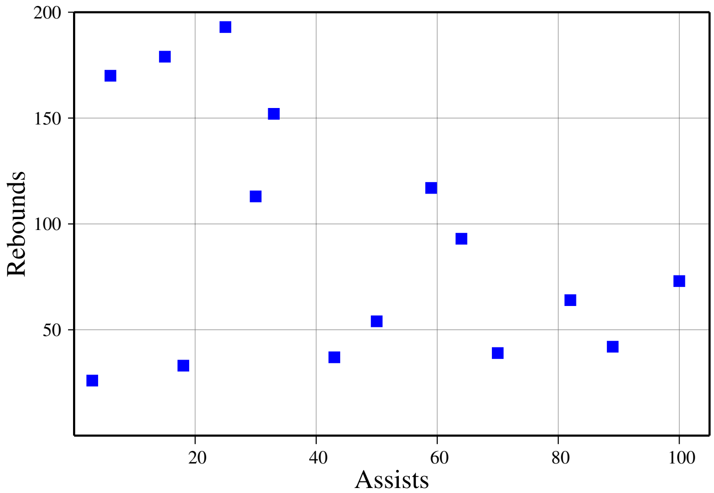


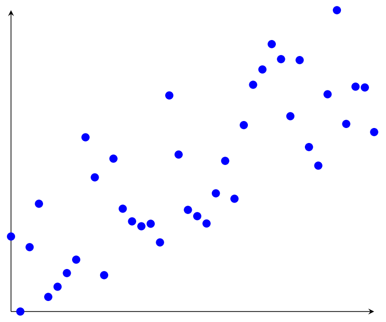
figure 2: 2012-2013 Girls Basketball Data

What is meant by *pattern of association*? It is perhaps easier to discuss what is NOT meant by this phrase. One cannot use statistics to argue whether or not change in one variable causes the other to change. As an over-used example, while there may be an association between the time of a rooster's crow and the time of sunrise, the rooster's crow certainly does not cause the sun to rise (although he might think it does). While the truth, in a particular context, may indeed be that "X causes Y", this conclusion cannot be drawn by statistical practices. The lack of ability to establish a cause-and-effect relationship between X and Y is precisely why we choose to use the word "association". More on this will be addressed later, via examples.

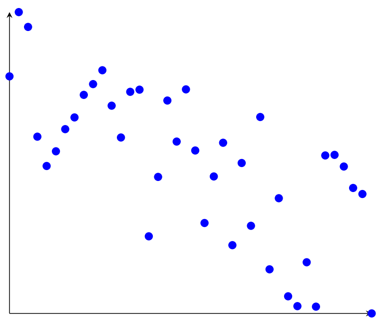
In the 8th grade, we focus on association in general, as well as linear association specifically (the latter more formally known as "correlation"). *Association* between two random variables refers to evidence of dependence, regardless of the nature of that dependence (e.g., linear, quadratic, or other). Loosely speaking, we call an association *positive* if Y generally increases as X increases, *negative* if Y generally decreases as X increases, and no (or zero) association if Y tends to remain the same regardless of changes in X. A linear association refers to an association that is well-captured by a linear relationship; exactly how that linear relationship is determined (i.e., how to sketch a "best-fit" line, and how "well" a line captures the behavior of bivariate data will be discussed in Section 6.3. A perfect linear association occurs if the data fall exactly on a line of either positive or negative slope. If the data fall perfectly on a horizontal line (zero slope), there is technically a linear (but uninteresting) association, as change in X does not influence Y .

Consider the following scatter plots.

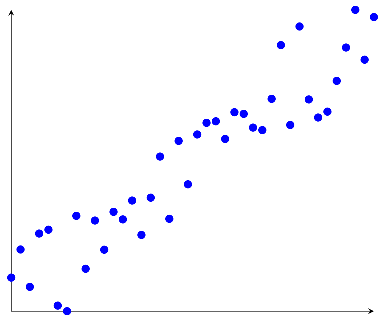
Positive Correlation



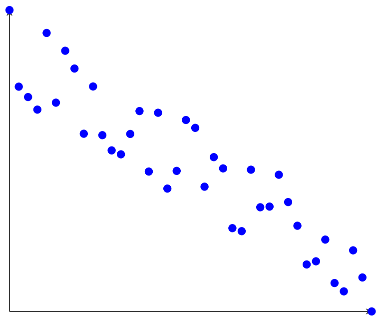
Negative Correlation



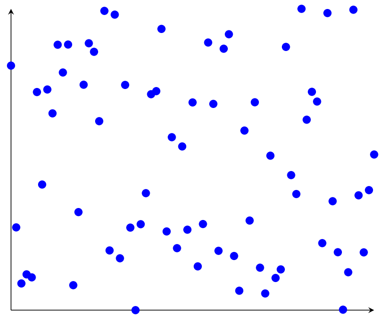
Strong Positive Correlation



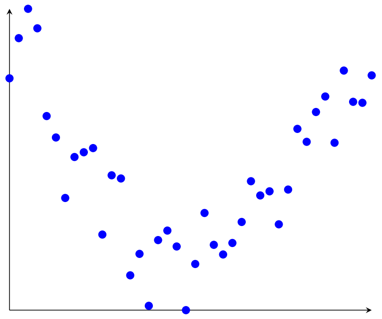
Strong Negative Correlation



No Correlation

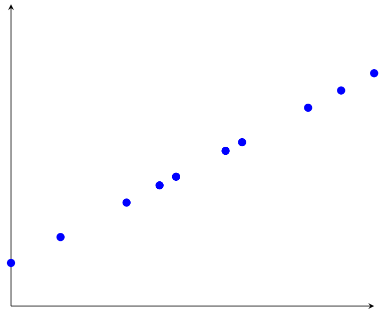


No Linear Correlation

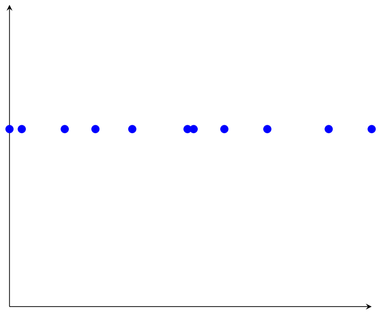


Perfect linear association occurs when all points fall upon the same line:

Perfect Linear Association



Perfect Linear Association (zero correlation)



While 8th grade students need only discuss "association", as opposed to "correlation", it is important for teachers and parents to know the difference, should the topic happen to arise. The above example of "perfect linear association" shows a positive correlation. However, contrast this example with the scatter plot on the right depicting points lying on a horizontal line. While the points have a perfect linear association, they have zero correlation, as Y is completely unaffected by the existence of X. Again, this needs not be discussed directly in the 8th grade, but teachers should have this distinction tucked away in the back of their minds.

Example 4.

> Recall the example of Izumi and her basketball data (return to figure 1). Upon studying the scatter plot of field goal data, we see that the data suggest a strong positive linear association between the number of field goals made and the number of field goals attempted over the course of a basketball season. Izumi thinks, "I suppose this positive association makes sense, because these players are pretty good at what they do; I would expect that more field goals attempted by these skilled players would be associated with more field goals made. Of course, as a side-note, this positive association may not hold true for unskilled basketball players, such as my cat, Mittens. No matter how many field goals Mittens attempts, she's probably not ever going to get the ball in the basket."
>
> Izumi then turns her attention to her Rebounds vs. Assists scatter plot (Fiigure 2), noticing a general negative linear association. She thinks, "This association seems weaker than that between the field goal variables, because these data points seem to be more scattered about the plane." After noting the negative association, Izumi thinks more deeply about what this might mean. "My scatter plot suggests that players who tend to have more assists also tend to get fewer rebounds. I guess this makes sense because players who tend to get assists are usually farther away from the basket, assisting to those players who tend to be closer to the basket. Of course! It's those players who tend to play closely to the basket who tend to get the rebounds."

Later, in Example 7, we discuss the important difference between association and causation. For now, note that an association between variables does not imply that changes in one variable *cause* changes in the other variable.

In addition to observing trends (i.e., linear or nonlinear associations) in bivariate data, 8th grade students are to describe other characteristics, such as clustering and outliers. *Outliers* are data points that notably deviate or "stand out" from the general behavior of the data set. In 6th grade students studied several techniques to locate such standouts; for bivariate data we make use of clustering.

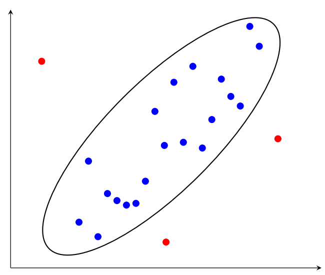


figure 3

In general, *clustering* refers to a set of data points that are in close proximity to each other. A single scatterplot may have many clusters of different sizes, and different clusters may be of different scientific interests. To exploit clustering for the purpose of identifying outliers, we consider the process of drawing a region around what seems like a self-consistent cluster of the data (see figure 3) for an example.

In this figure, even without the ellipse, it is clear that, except for three points, the data follow generally a positive linear trend. Often the data are not so convenient, and it takes some skill to identify outliers. In any case, whether easy or hard to identify, once the outliers have been identified graphically, the researcher still must give justification to treat them as outliers *in terms of the context*. Often, particularly in biomedical research, the outliers are the important data points (perhaps identifying risk factors in a new medication being tested). There are various tools that researchers can use to identify clusters and outliers (one is *projection pursuit*); but even when using those tools, the researcher must seek an explanation of outsider status (i.e., experimental error, an erratic member of the sample, origin of data suspect, etc.).

In figure 2 we see a definite negative trend in the data (rebounds decrease as assists increase), but the tendency follows a broad band rather than a line. Let's see how Izumi treats this:

Example 5.

Upon studying her Rebounds vs. Assists scatter plot (figure 2), Izumi concludes that the negative association would be stronger if not for the data point JM (the initials of Juanita Martinez), which lies rather strikingly outside from the main graphical trend. This particular datum is likely an outlier. Izumi thinks, "I wonder if there is some reason why Juanita's datum stands out from the rest of the team. Oh of course! Juanita transferred to my school from another one mid-season! Perhaps I should not include her in my data analysis." Here, Izumi has scientific reason to drop this outlier from her data set. Izumi will further investigate the ramifications of dropping Juanita from her analysis in the next section.

Note that Izumi has a reason, based on the circumstances, to identify Juanita as an outlier. Looking at the graph, she may also have considered Susan Garcia (SG) and Joan O'Connell (JO) as outliers, but not found a contextual reason to exclude them, so left them in the data analysis. In almost every such analysis, the identification of clusters and outliers (even if performed by a computer algorithm) has to be reinterpreted and justified in terms of the context. The important thing is that students are engaged in scientific discussions regarding various reasons for identifying various points as potential outliers. and consequently investigate dropping this particular datum. At the same time it should be stressed that there does not need to be an identifiable anomaly (e.g., some characteristic about the nature of the datum), such as Juanita being on the team for only part of the season, that logically sets it apart from the rest of the data) associated with a datum to label it an outlier. Outliers are simply data points that "stand apart from the general trend", regardless of the reason. So, the mathematics identifies "potential outliers," but the context explains why they should be excluded. Outliers should not be excluded simply because the mathematics has so identified them. For, an outlier may flag a confounding variable that is interfering with the relationship of interest. Or an outlier may just be an anomaly that should be ignored and dropped from the data analysis. Only an analysis in context can distinguish erroneous data points from critical pointers toward further research, and in fact, may not do so conclusively.

Section 6.2 Linear Models for Problem Solving

Construct and Assess Best Fitting Lines

Know that straight lines are widely used to model relationships between two quantitative variables. For scatter plots that suggest a linear association, informally fit a straight line, and informally assess the model fit by judging the closeness of the data points to the line. 8.SP.2

We deepen our understanding of an association of two variables by fitting (as best as possible) a straight line to the scatter plot of collected data. There are algorithms for determining (in some measure of distance between data sets) the "best fitting line," but here we will just eyeball the data. Return to the scatterplots on pages 5,6: the first four suggest a linear relation (the third and fourth more strongly than the first and second) and the fifth and sixth suggest that there is not a linear association. Just as a mean, median, or mode provides a single-point (zero-dimensional) description of an univariate data set (1-dimensional), a line provides a one-dimensional summary of a bivariate (2-dimensional) data set. Furthermore, just as there are various ways to "measure the center" of a one-variable data set, there are various ways to fit a line to bivariate data. Here, we explore a number of options, focusing on the "eye-balling" technique. Recommended eye-balling software include the Illuminations website from the National Council of Teachers of Mathematics:

`http://illuminations.nctm.org/ActivityDetail.aspx?ID=146`

This software has the feature that one can easily load data, and then can eyeball a best fitting line as well as ask for the calculation of "a best fitting line." You will be impressed how well the eye-ball guess is to the one created by formula. (There is a physical explanation of this: the eye reacts to the energy produced by the input, and the mathematical formula is based on a concept of energy between two sets of data).

If students want a method of fitting lines that is both accessible and consistent between individuals (as opposed to “eye-balling”, wherein each student’s line will slightly differ from those of others), the teacher may want to investigate the median-median line. The median-median line is accessible to 8th graders because they have previously learned the concept of median as a measure of center for univariate data (furthermore, the concept of median is reinforced). The Quantitative Literacy Series book *Exploring Data* (Dale Seymour Publications, 1986) provides an excellent explanation of this method. While “least squares” (the energy method) is the most widely-used technique by scientists, it is too computationally intensive for the 8th grade, which should be informal and exploratory. The median-median line provides a precise algorithm by which different students can obtain the same line, requiring only visual (as opposed to computational) techniques. It is a pedagogical tool, rather than a realistic tool.

The line that one fits to a scatter plot is meant to capture the behavior of the bivariate data, but in a simpler form than the entire plot. Using this line, one can make estimated predictions (e.g., interpolation and extrapolation) about the random variables X and Y , as they behave together.

Example 6.

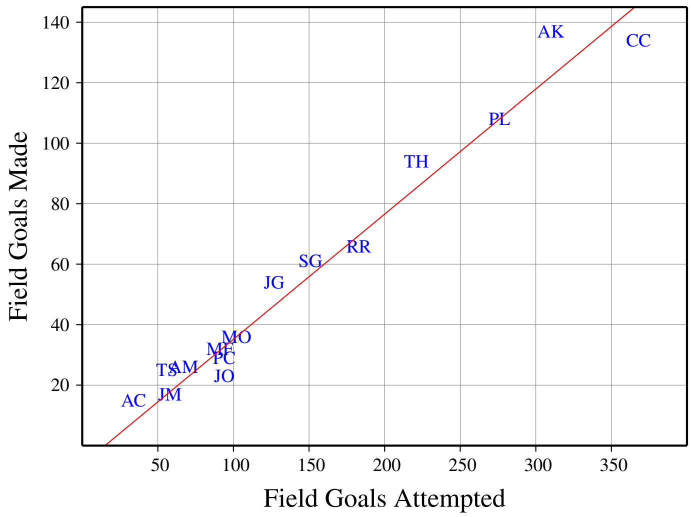


Figure 4: 2012-2013 Girls Basketball Data

To summarize the general behavior of her data, Izumi decides to draw a line to fit her scatter plots. Izumi first looks at the scatter plot of Field Goals Made vs. Field Goals Attempted (figure 1). Using her artistic skills, she adjusts her transparent ruler on the plot until she thinks she has found a line that is as close to each datum as possible. That is, Izumi tries her best to find the line that minimizes the distance between each data point and that line. After carefully changing the angle of her ruler, she decides to trace the red line shown in fFgure 4.

Next, Izumi turns to her Rebounds vs. Assists data, using her ruler to fit the red line as shown in figure 5.

Notice that the data points in figure 4 are more tightly clustered around Izumi’s best-fit line than the points in the Rebounds vs. Assists scatter plot (figure 5). In fact, figure 5 supports the earlier decision to exclude as an outlier the point denoted JM: If we ignore that point, clearly there is a better fitting best fitting line. Since the context supports the decision to consider JM as an outlier, now Izumi eyeballs a best fitting line with that point excluded, and generates figure 6.

Notice, as does Izumi, that the fit is better than in figure 5; in fact, Izumi created a composite, showing both lines (figure 7) to drive home this observation.

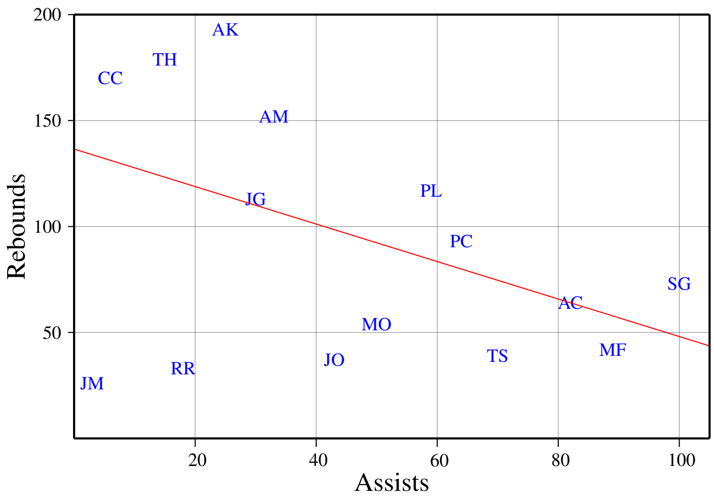


Figure 5: 2012-2013 Girls Basketball Data

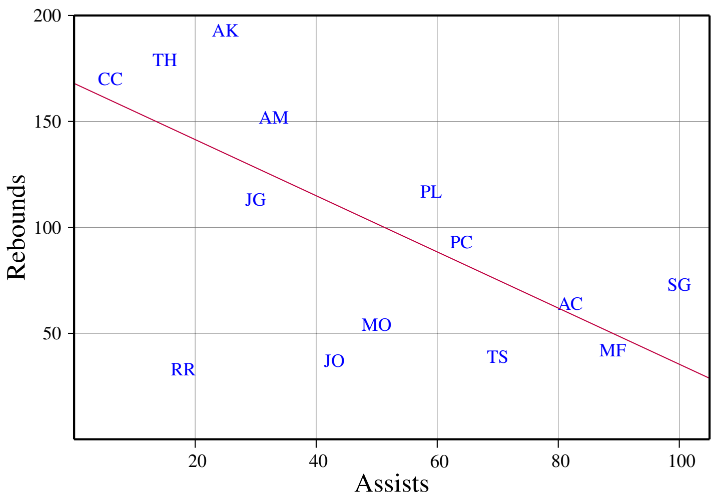


Figure 6: Girls Basketball Data, Outlier Removed

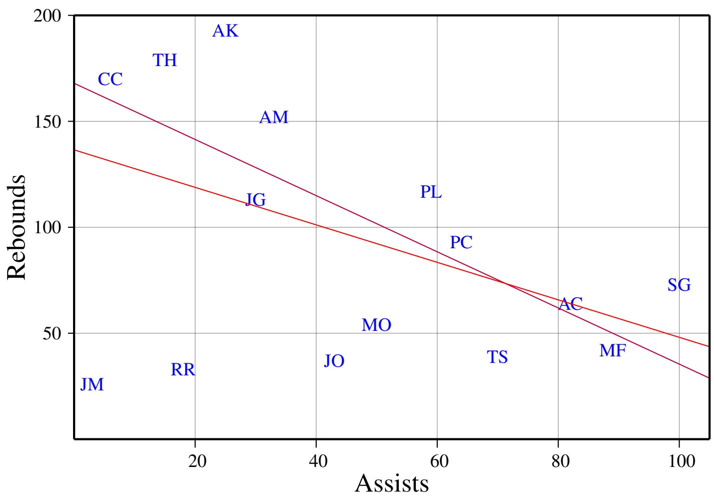


Figure 7: 2012-2013 Girls Basketball Data

Using Linear Models to Solve Problems

Use the equation of a linear model to solve problems in the context of bivariate measurement data, interpreting the slope and intercept.For example, in a linear model for a biology experiment, interpret a slope of 1.5 cm/hr as meaning that an additional hour of sunlight each day is associated with an additional 1.5 cm in mature plant height. 8.SP.3

In the preceding section we “modeled” bivariate data (X, Y) with a “best fitting line.” What can our line suggest to us, regarding the true relationship between random variables X and Y? First, let’s be careful about this question: we don’t know the true relation; we don’t even know if there is a relation. So the question really is: what does the best fitting line allow us to conclude about our relation, and can we justify those conclusions with scientific arguments. Let’s return to our good friend, Izumi, who is wrestling with these issues in a specific context.

Example 7.

Izumi notes that her lines of best-fit have slopes consistent with her original scatter plot assessments: The Field Goals Made vs. Field Goals Attempted plot suggested a positive linear association between her variables, and her line of best fit has a positive slope (see figure 4 above) ; likewise the negative association between the variables of her Rebound vs. Assists scatter plot is reflected by the negative slope of her best-fit line (see figure 7). Izumi realizes that she can calculate the slopes of her lines and make quantified statements about her variables.

As Izumi has already learned how to compute the slope of a line and locate the line’s y-intercept, she approximates the following equation for her field goal line: $Y = 0.41X - 3.4$. Izumi thinks about what this means and concludes, “My line of best fit suggests that approximately each additional shot that a basketball player at my school attempts during a game is associated with a roughly 41% increase in the number of her made field goals for the season. Of course, this doesn’t say anything about a given individual player; it just describes my data in general. It gives me a good guess at what I might expect from a random player, but I can’t be certain.” Izumi continues to think about her equation, focusing on the y-intercept of $(0, -3.4)$. “Hmm, speaking of making guesses, my line would predict that a random player who had zero field goal attempts would have made -3.4 of them! That’s ridiculous! It just goes to show that my linear model (a linear equation describing a relationship between variables, informed by observations) has limitations. While my model may make reasonable predictions for numbers near the range of my data, it may not make sense for extremes.”

As Izumi continues to think more deeply about what this association might be telling her, she realizes, “While it makes sense that a higher number of field goals made necessarily means that at least that number of field goals were attempted, it is interesting that, for any given person in general, the number of attempted field goals does not necessarily cause the number of made field goals to be a certain number. Instead, all that I can say is that my data suggest an association between the two variables. For example, anybody can throw the basketball in an attempt to make a field goal, but it’s the player’s skill (and perhaps the defense’s lack of skill?) that actually causes the ball to go into the basket.”

After patting herself on the back for recognizing the difference between causation and association, Izumi turns her attention to the Rebounds vs. Assists lines of best fit. She calculates the slopes and intercepts for both the red (including the outlier, Juanita Martinez) and the blue (omitting the outlier) lines, and calculates the equations of those lines.

“Interesting,” Izumi thinks. “If I omit the outlier, then my scatter plot suggests a stronger negative association between assists and rebounds than if I were to include the outlier. Specifically, if I include Juanita, then my linear model suggests that each additional assist is associated with a 92% decrease in that player’s number of rebounds. However, if I omit Juanita, then my linear model suggests that each additional assist is associated with a 140% decrease in that player’s rebounds! ”

After thinking about the slopes of her lines in the context of her experiment, Juanita begins to think about the y-intercepts of her linear models. "If my red line were a reasonable model describing the relationship between rebounds and assists, then a reasonable expectation to have regarding a player who made zero assists for the season would be that the player made 138 rebounds. If my blue line were a reasonable model, then I might venture the guess that a player who makes zero assists would have made 172 rebounds for the season. While these numbers are predicted by my respective linear models, the usefulness is not clear to me, because I've never known a player to have zero assists for an entire season. But it makes for an interesting thought! Come to think of it, if a player has zero assists, then that player likely warmed the bench a lot more than she actually played, so it seems more reasonable to guess that she would have very few rebounds. Gee, it sure is important to continue to think critically, engage my brain, and exploit my knowledge of basketball while I analyze my data!"

Section 6.3: Analyzing Bivariate Categorical Data Using Two-way Frequency Tables

Understand that patterns of association can also be seen in bivariate categorical data by displaying frequencies and relative frequencies in a two-way table. Construct and interpret a two-way table summarizing data on two categorical variables collected from the same subjects. Use relative frequencies calculated for rows or columns to describe possible association between the two variables.For example, collect data from students in your class on whether or not they have a curfew on school nights and whether or not they have assigned chores at home. Is there evidence that those who have a curfew also tend to have chores? 8. SP.4

In the previous sections, our random variables have been *quantitative*. Scatterplots provide a natural way of visualizing bivariate quantitative data, because each real-valued realization of each quantitative random variable can be plotted on a number line (and thus a real-valued ordered pair can be plotted in the Cartesian coordinate plane). In contrast, this section investigates patterns of association between *categorical* variables, which are characterized by their qualitative nature (recall Section 6.1). Scatterplots are not useful for categorical bivariate data since categorical data cannot necessarily be ordered on a number line in any meaningful way. For example, consider the categorical variable "reptiles of Washington county, Utah". A few realizations of this categorical variable include "Desert Tortoise", "Chuckwalla", and "Western Rattlesnake". How sensible is it to plot these variables on a number line, given that there is no natural "order" affiliated with them? In short, number lines are reserved for numbers, so we need a to take a new approach to analyzing categorical data.

Two-Way Frequency Tables

Before we consider visual representations of bivariate categorical data, we first discuss a convenient way of summarizing such data: The *two-way frequency table*. The table is "two-way" because each bivariate datum is composed of an ordered pair of realizations from two categorical random variables. For example, a datum might be something like the ordered pair (female, non-smoker), or perhaps (green eyes, brown hair), or maybe (8th grader, basketball). The table is a "frequency" table because the cell entries count the number of subjects (i.e., the frequency of data points) that fall into each combination of categories. Consider the following example.

Example 8.

The Utah Fish and Wildlife Service has collected data regarding the protective status ("endangered", "threatened", "candidate", and "proposed/petitioned") of various Utah species (mammals, birds, reptiles, fishes, insects, snails, and flowering plants) with which the Utah Ecological Services is concerned. As of April 2013, the agency reported the following data in

`www.fws.gov/utahfieldoffice/endspp.html`

organized in a two-way frequency table. The first two categories are the protected categories, the category "candidate" includes those species that the Service has decided to consider for explicit protection, and the category "proposed/petitioned" consists of species brought to the attention of the Service by other groups. These data include only species that have both a protective status and are of interest to the Utah Ecological Services Field Office, disregarding all Utah species that do not have such status.

	Endangered	Threatened	Candidate	Proposed
Mammal	1	2	0	1
Bird	2	1	2	1
Reptile	0	1	0	0
Fish	7	2	1	0
Insect	0	0	0	1
Snail	1	0	0	0
Flowering PLant	11	13	6	4

Here, each datum is an ordered pair realization of the bivariate categorical random variable of the form (species type, protective status), such as (fish, threatened). Each cell of this two-way frequency table displays the frequencies (counts) of each possible combination of variables that are observed. For example, there is one mammal species with an "endangered" status, two mammal species with a status of "threatened", and six flowering plant species that are "candidates" for being granted a protective status.

In general, a two-way frequency table is designed as follows:

<table>
<tr><td colspan="2" rowspan="2"></td><td colspan="4">Categorical Random Variable #2</td></tr>
<tr><td>Realization A of Variable #2</td><td>Realization B of Variable #2</td><td>Realization C of Variable #2</td><td>$\cdots$</td></tr>
<tr><td rowspan="4">Categorical Random Variable #1</td><td>Realization A of Variable #1</td><td>Frequency of (A_1, A_2)</td><td>Frequency of (A_1, B_2)</td><td>Frequency of (A_1, C_2)</td><td>$\cdots$</td></tr>
<tr><td>Realization B of Variable #1</td><td>Frequency of (B_1, A_2)</td><td>Frequency of (B_1, B_2)</td><td>Frequency of (B_1, C_2)</td><td>$\cdots$</td></tr>
<tr><td>Realization C of Variable #1</td><td>Frequency of (C_1, A_2)</td><td>Frequency of (C_1, B_2)</td><td>Frequency of (C_1, C_2)</td><td>$\cdots$</td></tr>
<tr><td>$\vdots$</td><td>$\vdots$</td><td>$\vdots$</td><td>$\vdots$</td><td>$\ddots$</td></tr>
</table>

We next give a simpler, fictional example to demonstrate the power of the two-way frequency table in the 8th grade setting.

Example 9.

Carlos enjoys spending time with his friends. He feels sad when one of his friends cannot hang out with him. Often when a friend cannot hang out, it is because the friend either cannot stay out late at night, or the friend is busy doing chores at home. Carlos notices that it tends to be the same group of friends who have curfews on school nights who also have chores to do at home. He wonders, "Do students at my school, in general, who have chores to do at home tend to also have curfews at night?"

Carlos decides to conduct an experiment to help suggest an answer to his question. He randomly surveys 52 students at his school, asking each student if s/he has a curfew and if s/he has to do household chores. To review some vocabulary, notice that Carlos's *experiment* is to record the responses of the 52 randomly selected students; the two *categorical random variables* of interest are "curfew status" (which has realizations "has curfew" and "does not have curfew") and "chores status" (which has realizations "has chores" and "does not have chores"). It is crucial that every subject in Carlos's study falls into exactly one category of each variable, that is, one cannot both have chores and not have chores.

Carlos observes that of the 52 students he surveyed, 31 have curfews and 35 have chores to do. Of the 31 students who have curfews, 26 also have chores to do. He summarizes the breakdown of his data in the following two-way frequency table:

	Has Curfew	No Curfew
Has Chores	26	9
No Chores	5	12

Notice how Carlos organizes his information: The realizations of the "curfew status" variable are the columns of the table; the realizations of the "chores status" variable are the rows of the table. Also notice that we can calculate the **marginal frequencies** (the count of the occurrence of one variable at a time).

	Has Curfew	No Curfew	
Has Chores	26	9	**35**
No Chores	5	12	**17**
	31	**21**	**52**

We see explicitly that there are 35 total students surveyed to have chores and there are 17 total who have no chores, as we take the total across the rows. Similarly, we see that there are 31 total students with curfews and 21 without curfews. Furthermore, note that the sums of each set of marginal frequencies must equal the total number of students surveyed: 35 + 17 = 52 and, likewise, 31 + 21 = 52.

It is always the case that the sum the marginal frequencies of a given variable equals the total number of subjects, so adding marginal frequencies provides a useful check for mistakes. As we will soon see, marginal frequencies help us answer important questions about our data. Let's get one more example under our belt before moving on to the interpretation of two-way frequency tables.

Example 10.

Emina loves to eat tomatoes from her garden in Salt Lake City. She asked her friend Renzo, "Don't you just love tomatoes?" Renzo crinkled his nose and replied, "Ew, tomatoes gross me out! When I see them in the grocery store, I just keep on walking." Renzo's response prompted Emina to think, "I don't buy tomatoes at the grocery store either, because I grow them in my garden. The tomatoes from my garden are delicious, whereas grocery story tomatoes look less appealing to me. I wonder if there is an association between enjoying tomatoes and having a garden at home."

Emina surveys 100 randomly-selected Salt Lake City vegetable-eating residents and asks each of them two questions: 1. Do you primarily obtain your vegetables at the grocery store (including food pantry), the farmer's market, or your home garden? 2. Do you like tomatoes? Emina summarizes her results in the following table:

	Grocery Store	Farmer's Market	Home Garden
Likes Tomatoes	50	4	12
Dislikes Tomatoes	30	1	3

Emina wonders if her data suggest an association between enjoying tomatoes and having a garden, but she's not yet sure how to use her data to investigate this question.

Making and Interpreting Two-Way Relative Frequency Tables

In this section, we transform our frequency tables into *relative* frequency tables, which often help us interpret data. A *relative frequency* refers to the ratio of the frequency of a particular realization of a bivariate categorical variable to the total number of observations. In other words, a relative frequency is a number between 0 and 1 (inclusive), commonly represented by a fraction, decimal, or percent. As a result, relative frequencies are useful in discussions of probabilities and thus interpretations of bivariate categorical data. We explain further by example, beginning with the construction of relative frequency tables, followed by their interpretation.

EXAMPLE 11.

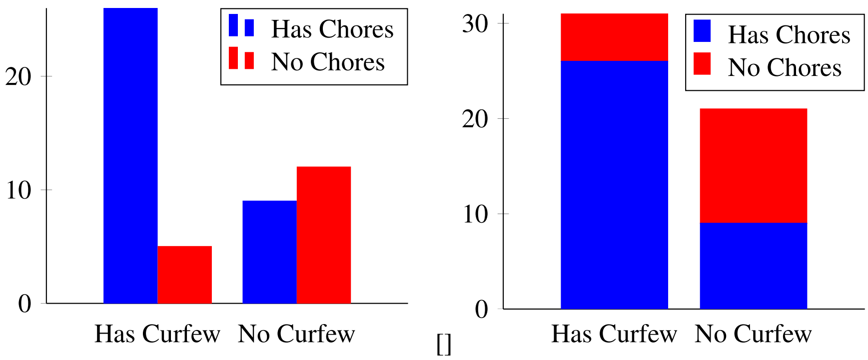
Recall Carlos's data regarding chores and curfew, specifically his two-way frequency table containing the marginal frequencies, copied below:

Frequency Table	Has Curfew	No Curfew	
Has Chores	26	9	**35**
No Chores	5	12	**17**
	31	**21**	**52**

The relative frequency table below was constructed from the table above.

Frequency Table	Has Curfew	No Curfew	
Has Chores	$\frac{26}{52} = 0.50$	$\frac{9}{52} \approx 0.17$	$0.05 + 0.17 \approx 0.67$
No Chores	$\frac{5}{52} \approx 0.096$	$\frac{12}{52} \approx 0.23$	$0.096 + 0.23 \approx 0.33$
	$0.50 + 0.096 \approx 0.06$	$0.17 + 0.23 = 0.40$	**1**

Note further that each pair of marginal relative frequencies necessarily have a sum of 1. Carlos can use his relative frequency table to draw conclusions such as, "Of the 52 randomly-selected students I surveyed, 67% of them have chores assigned to them at home, and about 60% of the students surveyed have a curfew." Carlos continues, "Perhaps the most striking observation to be made is that the bulk of students I surveyed (50%) fall into the category of both "Has Curfew" and "Has Chores"; the second-most-popular category is both "No Curfew" and "No Chores". This is interesting because it suggests an association between having a curfew and also having chores to do at home. That is, my survey suggests that students who have curfews also tend to have chores assigned to them." Carlos then used his relative frequency table to construct visual representations of his data, shown below (one with categories side-by-side, the other stacked).



[] []

Carlos's Data

Carlos constructed these graphs so that 50% of the cumulative bar area would indicate data falling under the "Has-chores-and-Has-Curfew" category, 23% would fall under the "No-Chores-and-No- Curfew" category, 17% would fall under the "Has-Chores-but-No-Curfew" category, and about 10% would fall under the "No-Chores-but-Has-Curfew" category. Such graphical representations often make it easy to visually inspect associations between variables. Since the vast majority of the "Has Curfew" bar is darkly shaded (indicating these subject also have chores), while the majority of the "No Curfew" bar is lightly shaded (indicating subjects who do not also have chores), the association is visually depicted.

EXAMPLE 12.

Recall the Utah Fish and Wildlife Service data from Example 8. To help us create a two-way relative frequency table, we again first include the marginal frequencies to our original frequency table.

Frequency Table	Endangered	Threatened	Candidate	Proposed	
Mammal	1	2	0	1	**4**
Bird	2	1	2	1	**6**
Reptile	0	1	0	0	**1**
Fish	7	2	1	0	**10**
Insect	0	0	0	1	**1**
Snail	1	0	0	0	**1**
Flowering Plant	11	13	6	4	**34**
	22	**19**	**9**	**7**	**57**

Check that the following relative frequency table can be constructed from the above two-way frequency table. Note that the relative frequencies can be expressed in fraction, decimal, or percent form, which provides an opportunity for students to review and practice such concepts.

Realative Frequency Table	Endangered	Threatened	Candidate	Proposed	
Mammal	1/67	2/57	0	1/57	**4/57**
Bird	2/57	1/57	2/57	1/57	**6/57**
Reptile	0	1/57	0	0	**1/57**
Fish	7/57	2/57	1/57	0	**10/57**
Insect	0	0	0	1/57	**1/57**
Snail	1/57	0	0	0	**1/57**
Flowering Plant	11/57	13/57	6/57	4/57	**34/57**
	22/57	**19/57**	**9/57**	**7/57**	**57/57=1**

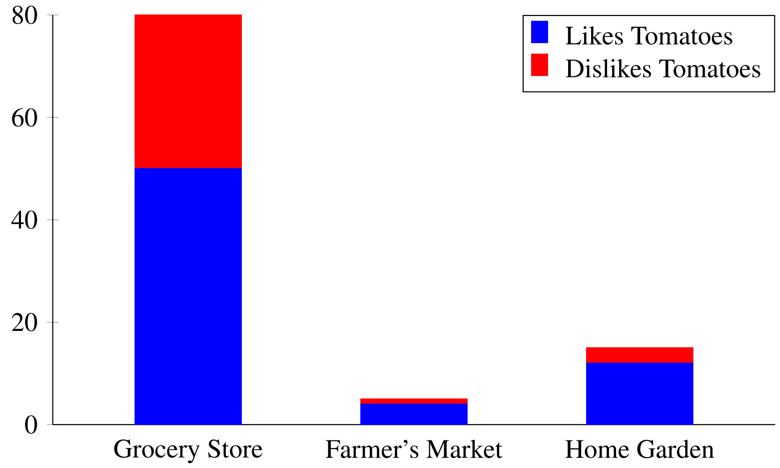
From the above chart, we can easily answer questions such as, “What percent of species with protective status in Utah are mammals?” Here, the marginal relative frequency of 4/57 tells us that only about 7% are mammals. One may ask why there are relatively few animals (mammals through snails) given protective status (about 40% of those with status) than flowering plants (about 60%). Perhaps the Fish and Wildlife Service has more of an incentive to classify plants than animals? Perhaps it is plants about which it is easier to collect data than animals which scurry about? Other questions one may be inspired to ask about these data may come from noting that, of the animals with protective status, the fish and birds far outnumber the snails and insects (16/57 and 2/57, respectively). Perhaps the Fish and Wildlife Service is more concerned with species of recreational interest (e.g., fishing and hunting)? Or perhaps there are biological or ecological reasons for the discrepancies?

EXAMPLE 13.

Recall the fictional Emina and her tomato garden (Example 6.5.1c). Emina summarizes her data in the following relative frequency table and stacked bar graphs.

Frequency Table	Grocery Store	Farmer’s Market	Home Garden	
Likes Tomatoes	0.50	0.04	0.12	**0.66**
Dislikes Tomatoes	0.30	0.01	0.03	**0.34**
	0.80	**0.05**	**0.15**	**1.00**

Emina quickly sees from her relative frequency table that the majority (80%) of the vegetable-eating people she surveyed purchase most of their veggies at a grocery store, and that only 15% of those surveyed mostly eat veggies from their gardens. “What’s most interesting to me,” thinks Emina, “is that even though a small percentage of people surveyed use their gardens as their main vegetable source, of



Eminia's Data

those 15%, a whopping 12 out of 15 people like tomatoes! That is, of those who have a home garden as their main veggie source, 80% (12/15) of them like tomatoes. This is a stark contrast with the grocery-shoppers: Of the 80% of people surveyed who buy most of their veggies at the grocery store, only 50 out of 80 like tomatoes, or just 62.5%. So it looks like there could be a positive association between having a home garden and liking the taste of tomatoes. I wonder if this means tomatoes are tastier out of a home garden than the store. Maybe I should offer Renzo a tomato from my garden...." Emina continues to think about her study results, and notices that 4 out of 5 people (also 80%) who obtain most of their veggies from the farmer's market also enjoy tomatoes. "Hmm. Eating tomatoes from a farmer's market is very similar to eating tomatoes out from a home garden, since the farmer's market produce is grown locally. Maybe I should pool these data together, since they're arguably telling me the same information about locally grown food." Emina continues to think deeply about her data, and after making the following graphs, concludes "Regardless, the ratios of darkly shaded (likes tomatoes) to lightly shaded (dislikes tomatoes) areas of individual bars on my stacked bar graph indicates that there is a positive association between locally grown produce and the enjoyment of tomatoes."

Chapter 7
Rational and Irrational Numbers

In this chapter we first review the real line model for numbers, as discussed in Chapter 2 of seventh grade, by recalling how the integers and then the rational numbers are associated to points in the line. Having associated a point on the real line to every rational number, we ask the question, do all points correspond to a rational number? Recall that a point on the line is identified with the length of the line segment from the origin to that point (which is negative if the point is to the left of the origin). Through constructions (given by "tilted" squares), we make an observation first made by the Pythagorean society 2500 years ago that there are lengths (such as the diagonal of a square with side length 1) that do not correspond to a rational number. The construction produces numbers whose squares are integers; leading us to introduce the symbol $\sqrt{A}$ to represent a number whose square is A. We also introduce the cube root $\sqrt[3]{V}$ to represent the side length of a cube whose volume is V. The technique of tilted squares provides an opportunity to observe the Pythagorean theorem: $a^2 + b^2 = c^2$, where a and b are the lengths of the legs of a right triangle, and c is the length of the hypotenuse.

In the next section we return to the construction of a square of area 2, and show that its side length ($\sqrt{2}$) cannot be equal to a fraction, so its length is not a rational number. We call such a number an *irrational number*. The same argument works for $\sqrt{5}$ and other lengths constructed by tilted squares. It is a fact that if N is a whole number, either it is a perfect square (the square of an integer), or $\sqrt{N}$ is not a quotient of integers; that is $\sqrt{N}$ is an *irrational number*.

In the next section we turn to the question: can we represent lengths that are not quotients of integers, somehow by numbers? The ancient Greeks were not able to do this, due mostly to the lack of an appropriate system of expressing lengths by their numerical measure. For us today, this effective system is that of the decimal representation of numbers (reviewed in Chapter 1 of seventh grade).

We recall from grade 7 that a rational number is represented by a terminating decimal only if the denominator is a product of twos and fives. Thus many rational numbers (like 1/3,1/7, 1/12,...) are not represented by terminating decimals, but they are represented by repeating decimals, and similarly, repeating decimals represent rational numbers. We now view the decimal expansion of a number as providing an algorithm for getting as close as we please to its representing point on the line through repeated subdivisions by tenths. In fact, every decimal expansion represents a point on the line, and thus a number, and unless the decimal expansion is terminating or repeating, it is irrational.

The question now becomes: can we represent all lengths by decimal expansions? We start with square roots, and illustrate Newton's method for approximating square roots: Start with some reasonable estimate, and follow with the recursion

$$a_{\text{new}} = \frac{1}{2}\left(a_{\text{old}} + \frac{N}{a_{\text{old}}}\right) .$$

Through examples, we see that this method produces the decimal expansion of the square root of N to any required degree of accuracy. Finally, we point out that to do arithmetic operations with irrational as well as rational numbers, we have to be careful: to get within a specified number of decimal points of accuracy we may need much better accuracy for the original numbers.

Section 7.1. Representing Numbers Geometrically

First, let us recall how to represent the rational number system by points on a line. With a straight edge, draw a horizontal line. Given any two points a and b on the line, we say that $a < b$ if a is to the left of b. The piece of the line between a and b is called the *interval between a and b*. It is important to notice that for two different points a and b we must have either $a < b$ or $b < a$. Also, recall that if $a < b$ we may also write this as $b > a$.

Pick a point on a horizontal line, mark it and call it the origin, denoted by 0. Now place a ruler with its left end at 0. Pick another point (this may be the 1 cm or 1 in point on the ruler) to the right of 0 and denote it as 1.We also say that the length of the interval between 0 and 1 is one (per *one unit*). Mark the same distance to the right of 1, and designate that endpoint as 2. Continuing on in this way we san associate to each positive integer a point on the line. Now mark off a succession of equally spaced points on the line that lie to the left of 0 and denote them consecutively as $-1,\ -2,\ -3,\ \ldots$. In this way we can imagine all integers placed on the line.

We can associate a half integer to the midpoint of any interval; so that the midpoint of the interval between 3 and 4 is 3.5, and the midpoint of the interval between -7 and -6 is -6.5. If we divide the unit interval into three equal parts, then the first part is a length corresponding to 1/3, the first and second parts correspond to 2/3, and indeed, for any integer p, by putting p copies end to and on the real line (on the right of the origin is $p > 0$, and on the left if $p < 0$), we get to the length representing $p/3$. We can replace 3 by any positive integer q, by constructing a length which is one qth of the unit interval. In this way we can identify every rational number p/q with a point on the horizontal line, to the left of the origin if p/q is negative, and to the right if positive.

The number line provides a concrete way to visualize the decimal expansion of a number. Given, say, a positive number a, there is an integer N such that $N \leq a < N + 1$. This N is called the *integral part* of a. If $N = a$, we are done. If not, divide the interval between N and $N + 1$ into ten equal parts, and let d_1 be the number of parts that fit in the interval between N and a. This d_1 is a *digit* (an integer between (and possibly one of) 0 and 9). This is the *tenths part* of a, and is written $N.d_1$. If $N.d_1 = a$, we are through. If not repeat the process: divide the interval between $N.d_1$ and $N.(d_1 + 1)$ into ten equal parts, and let d_2 be the number of these parts that fit between $N.d_1$ and a. This is the *hundredths part* of a, denoted $N.d_1d_2$. Continuing in this way, we discover an increasing sequence of numerical expressions of the form $N.d_1d_2\cdots$ that get closer and closer to a, thus providing an effective procedure for approximating the number a. As we have seen in grade 7, this process may never (meaning "in a finite number of steps") reach a, as is the case for $a = 1/3, 1/7$ and so on.

Now, instead of looking at this as a procedure to associate a decimal to a number, look at it as a procedure to associate a decimal to a length on the number line. Now let a be any point on the number line (say, a positive point). The same process associates a decimal expansion to a, meaning an effective way of approximating the length of the interval from 0 to a by a number of tenths of tenths of tenths (and so on) of the unit interval. We begin this discussion by examining lengths that can be geometrically constructed; leading us to an answer to the question: are there lengths that cannot be represented by a rational number? To do this, we need to move from the numerical representation of the line to the numerical representation of the plane.

Using the number line created above, draw a perpendicular (vertical) line through the origin and use the same procedure as above with the same unit interval. Now, to every pair of rational numbers (a, b) we can associate a point in the plane: Go along the horizontal (the x-axis) to the point a. Next, go a (directed) length b along the vertical line through a . This is the point (a, b).

Example 1.

In Figure 1 the unit lengths are half an inch each. To the nearest tenth (hundredth) of an inch, approximate the lengths $AB,\ AC,\ BC$.

Solution. We use a ruler to approximate the lengths. There will be a scale issue: the length of a side of a box may not measure half an inch on our ruler. After calculating the change of scale, one should find the length of AC to be about 3.05 in.

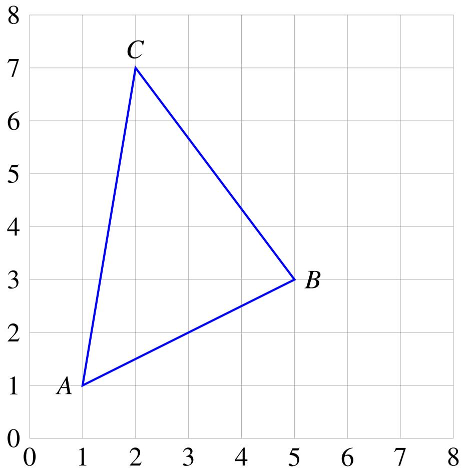


Figure 1

Now, it is important to know that, by using a ruler we can always estimate the length of a line segment by a fraction (a rational number), and the accuracy of the estimate depends upon the detail of our ruler. The question we now want to raise is this: can any length be described or named by a rational number?

The coordinate system on the plane provides us with the ability to assign lengths to line segments. Let us review this through a few examples.

Example 2.

In Figure 2 we have drawn a tilted square (dashed sides) within a horizontal square. If each of the small squares bounded in a solid line is a unit square (the side length is one unit), then the area of the entire figure is $2 \times 2 = 4$ square units. The dashed, tilted square is composed of precisely half (in area) of each of the unit squares, since each of the triangles outside the tilted square corresponds to a triangle inside the tilted square. Thus the tilted square has area 2 square units. Since the area of a square is the square of the length of a side, the length of each dashed line is a number whose square is 2, denoted $\sqrt{2}$.

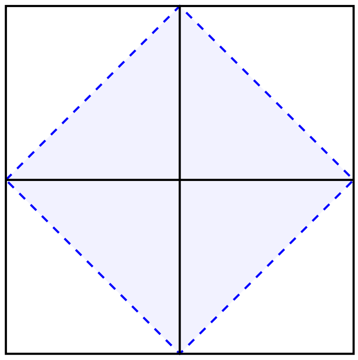


Figure 2

We will use this symbol $\sqrt{A}$ (*square root*) to indicate a number a whose square is A: $a^2 = A$. Since the square of any nonzero number is positive (and $\sqrt{0} = 0$), $\sqrt{A}$ makes sense only if A is not negative. Since $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, $5^2 = 25$, the integers 4, 9, 16, 25 have integers as square roots. A positive integer whose square root s a positive integer is called a *perfect square*. For other numbers (such as 2, 3, 5, 6, . . .), we still have to find a way to calculate the square roots. This strategy, of tilted squares, gives a way of constructing lengths corresponding to square roots of all whole numbers, as we shall now illustrate.

Example 3.

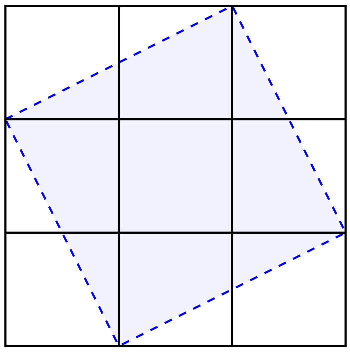


Figure 3

In Figure 3 the large square has side length 3 units, and thus area of 9 square units. Each of the triangles outside the tilted square is a 1×2 right triangle, so is of area 1. Thus the area of the tilted square is $9 - 4 = 5$, and the length of the sides of the tilted square is $\sqrt{5}$.

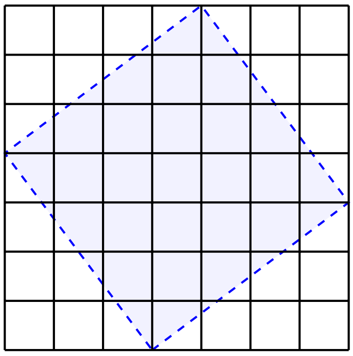


Figure 4

By the same reasoning: the large square in Figure 4 is 7×7 so has area 49 square units. Each triangle outside the tilted square is a right triangle of leg lengths 3 and 4, so has area 6 square units. Since there are four of these triangles, this accounts for 24 square units, and thus the area of the tilted triangle is $49 - 24 = 25$ square units. Since $25 = 5^2$, the side of the tilted square has length 5 units. That is, $\sqrt{25} = 5$.

Example 4.

As we shall see in more detail, these examples generalize (with a little ingenuity) to a formula for the length of the hypotenuse of a right triangle, given the lengths of its legs (this is the *Pythagorean Theorem*). Here we demonstrate this formula using tilted squares. For any two lengths a and b, draw a square of side length $a + b$ as shown in Figure 5. Now draw the dashed lines as shown in that figure.

The figure bounded by the dashed lines is a square; denote its side length by c. Then the area of the square is c^2. Each of the triangles is a right triangle of leg lengths a and b and hypotenuse length c. Now, move to Figure 6 which isubdivides the $a + b$-sided square in a different configuration: the bottom left corner is filled with a square of side length b, and the upper right corner, by a square of side length a. The rest of the big square of Figure 6 is a pair of congruent rectangles. By drawing in the diagonals of those rectangles as shown, we see that this divides the two rectangles into four triangles, all congruent to the four triangles of Figure 5. Thus what lies outside these triangles in both figures has the same area. But what remains in Figure 5 is a square of area c^2, and what remains in Figure 6 are the two squares of areas a^2 and b^2. This result is:

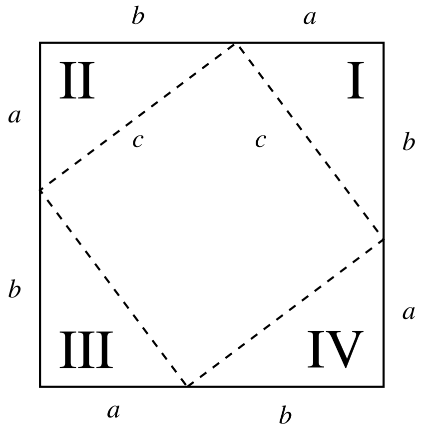


Figure 5

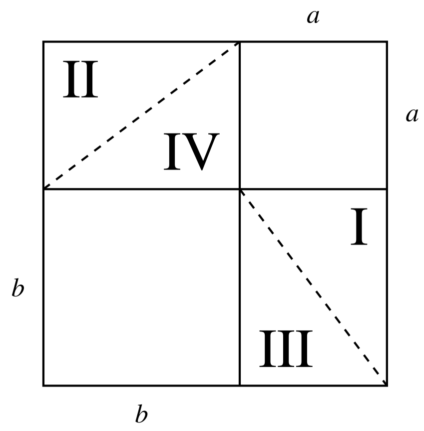


Figure 6

The Pythagorean Theorem:

$$a^2 + b^2 = c^2$$

for a right triangle whose leg lengths are a and b and whose hypotenuse is of length c.

This theorem allows us to find the lengths of the sides of tilted squares algebraically. For example, the tilted square in Figure 1 has side length c where c is the length of the hypotenuse of a right triangle whose leg lengths are both 1:

$$c^2 = 1^2 + 1^2 = 2\ ,$$

so $c = \sqrt{2}$. Similarly for Figure 2: $c^2 = 1^2 + 2^2 = 5$, so $c = \sqrt{5}$. For Figure 3, we calculate $c^2 = 3^2 + 4^2 = 9 + 16 = 25$, so $c = \sqrt{25} = 5$.

Section 7.2 Solutions to Equations Using Square and Cube Roots

Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. 8.EE.2, first part.

In the preceding section we introduced the symbol $\sqrt{A}$ to designate the length of a side of a square of area A. Similarly, the side length of a cube of volume V as $\sqrt[3]{V}$ (called the *cube root* of V). Another way of saying this is that $\sqrt{A}$ is the solution of the equation $x^2 = A$ and $\sqrt[3]{V}$ is the solution of the equation $x^3 = V$. We also defined a perfect square as an integer whose square root is an integer. Similarly an integer whose cube root is an integer is a *perfect cube*. Here is a table of the first few perfect squares and cubes.

Number	1	2	3	4	5	6	7	8	9
Square	1	4	9	16	25	36	49	64	81
Cube	1	8	27	64	125	216	343	512	729

For numbers that are not perfect squares, we can sometimes use factorization to express the square root more simply.

Example 5.

a. Since 100 is the square of ten, we can write $\sqrt{100} = 10$. But we could also first factor 100 as the product of two perfect squares: $\sqrt{100} = \sqrt{4 \cdot 25} = \sqrt{4} \cdot \sqrt{25} = 2 \cdot 5 = 10$.

b. Similarly, $\sqrt{729} = \sqrt{9 \cdot 81} = \sqrt{9} \cdot \sqrt{81} = 3 \cdot 9 = 27$. Also, since $729 = 27^2$, and $3^3 = 27$,

$$\sqrt[3]{729} = \sqrt[3]{27 \cdot 27} = \sqrt[3]{27} \cdot \sqrt[3]{27} = 3 \cdot 3 = 9 .$$

c. Of course not every number is a perfect square, but we still may be able to simplify: $\sqrt{72} = \sqrt{36 \cdot 2} = \sqrt{36} \cdot \sqrt{2} = 6\sqrt{2}$.

d. $\sqrt{32} = \sqrt{4 \cdot 4 \cdot 2} = 4\sqrt{2}$.

Similarly, we can try to simplify arithmetic operations with square roots:

e. $\sqrt{6}\sqrt{12} = \sqrt{6}\sqrt{6 \cdot 2} = \sqrt{6}\sqrt{6}\sqrt{2} = 6\sqrt{2}$.

f. $\sqrt{2} + \sqrt{8} = \sqrt{2} + \sqrt{4}\sqrt{2} = \sqrt{2} + 2\sqrt{2} = 3\sqrt{2}$.

The following example goes beyond the eighth grade standards, but is included to emphasize to students that the rules of arithmetic extend to expressions with root symbols in them.

Example 6.

a. Solve: $3\sqrt{x} = 39$.

Solution. Dividing both sides by 3, we have the equation $\sqrt{x} = 13$. Now squaring both sides, we get $x = 13^2 = 169$.

b. Solve: $5\sqrt{x} + \sqrt{x} = 24$.

Solution. $5\sqrt{x} + \sqrt{x} = 6\sqrt{x}$,, leading to the equation $6\sqrt{x} = 24$,simplifying to $\sqrt{x} = 4$. Squaring, we get the answer: $x = 4^2 = 16$.

c. Which is greater: 14 or $10\sqrt{2}$?

Solution. We want to determine if the statement $14 < 10\sqrt{2}$ is true or false. We could approximate $\sqrt{2}$, but the truth of our answer depends upon the accuracy of our approximation. A better strategy is to square both sides, since we know that squaring two positive numbers preserves the relation between them. So our test becomes: is this true: $14^2 < 100(2)$. Since $14^2 = 196$, the answer is: yes, this is true.

d. Is there an integer between $4\sqrt{3}$ and $5\sqrt{2}$?

Solution. To solve this, we need to square both sides and ask: is there a perfect square between these numbers. Now $(4\sqrt{3})^2 = 48$ and $(5\sqrt{2}) = 50$, and the only integer between these two is 49. Since $49 = 7^2$, the answer is "yes: 7".

Section 7.3. Rational and Irrational Numbers

The Rational Number System

Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers, show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.8. NS. 1.

Know that $\sqrt{2}$ *is irrational. 8. EE. 2, second part.*

Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g.,π^2*). For example, by truncating the decimal expansion of* $\sqrt{2}$*, show that* $\sqrt{2}$ *is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations.8.NS.2*

The discussion about the relationship of numbers and lengths (summarized at the beginning of section 7.1), and their representation as decimals was a significant part of seventh grade mathematics. We now summarize that discussion. The decimal representation of rational numbers is the natural extension of the base ten place-value representation of whole numbers. Decimals are constructed by placing a dot, called a *decimal point*, after the units' digit and letting the digits to the right of the dot denote the number of tenths, hundredths, thousandths, and so on. If there is no whole number part in a given numeral, a 0 is usually placed before the decimal point (for example 0.75).

Thus, a terminating decimal is another representation of a fraction whose denominator is not given explicitly, but is understood to be an integer power of ten. Decimal fractions are expressed using decimal notation in which the implied denominator is determined by the number of digits to the right of the decimal point. Thus for 0.75 the numerator is 75 and the implied denominator is 10 to the second power (100), because there are two digits to the right of the decimal separator. In decimal numbers greater than 1, such as 2.75, the fractional part of the number is expressed by the digits to the right of the decimal value, again with the value of .75, and can be expressed in a variety of ways. For example,

$$\frac{3}{4} = \frac{75}{100} = \frac{7}{10} + \frac{5}{100} = 0.75\ ,$$
$$\frac{11}{4} = 2\frac{3}{4} = 2 + \frac{75}{100} = 2 + \frac{7}{10} + \frac{5}{100} = 2.75\ .$$

In seventh grade we observed that the decimal expansion of a rational number always either terminates after a finite number of digits or eventually begins to repeat the same finite sequence of digits over and over. Conversely such a decimal represents a rational number. First let's look at the case of terminating decimals.

Example 7.

Convert 0.275 to a fraction.

Solution.

$$0.275 = \frac{2}{10} + \frac{7}{100} + \frac{5}{1000} = \frac{2}{10} + \frac{7}{10^2} + \frac{5}{10^3}\ .$$

Now, if we put these terms over a common denominator, we get

$$\frac{2(10^2) + 7(10) + 5}{10^3} = \frac{275}{10^3}\ .$$

In general, a terminating decimal is a sum of fractions, all of whose denominators are powers of 10. By multiplying each term by 10/10 as many times as necessary, we can put all terms over the same denominator. In the same way, 0.67321 becomes

$$\frac{67321}{10^5},$$

and

$$0.0038 = \frac{3}{1000} + \frac{8}{10000} = \frac{38}{10000}.$$

So, we see that a terminating decimal leads to a fraction of the form $A/10^e$ where A is an integer and e is a positive integer. But we can make an even stronger statement, as follows

Notice that the expression $A/10^e$ is not necessary in lowest terms:

$$\frac{275}{10^3} = \frac{5 \cdot 5 \cdot 11}{2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5} = \frac{11}{2\dot{2} \cdot 2 \cdot 5} = \frac{11}{40},$$

and all we can say about this denominator is that it a product of 2's and 5's. But this is enough to guarantee that the fraction has a terminating decimal representation. In general, when $A/10^e$ is put into simplest terms, the denominator will still be a product of 2's and/or 5's so the fraction must have a terminating decimal representation.

Example 8.

a) $25 = 5^2$, so we should expect that 1/25 can be represented by a terminating decimal. In fact, if we multiply by 1 in its disguise as $\frac{2\times2}{2\times2}$ we get:

$$\frac{1}{25} = \frac{4}{100} = 0.04.$$

b) Consider $1/200$. Since $200 = 2^3 \cdot 5^2$, we lack a factor of 5 in the denominator to have a power of 10. We fix this by multiplying the fraction by 1 in the form $\frac{5}{5}$ to get $5/1000 = .005$.

In short, a fraction p/q can be written as a terminating decimal if q is a factor of a power of 10. Otherwise put

> A terminating decimal leads to a fraction whose denominator is a product of 2's and/or 5's, and conversely, any such fraction is represented by a terminating decimal.

What about a fraction of the form p/q, where q is not a product of 2's and/or 5's? In sixth grade we learned that by long division (of p by q) we can create a decimal expansion for p/q to as many places as we please. In seventh grade we went a little further. Since each step in the long division produces a remainder that is an integer less than q, after at most q steps we must repeat a remainder already seen. From that point on each digit of the long division repeats, and the process continues indefinitely in this way. For example, $1/3 = 0.3333\ldots$ for as long as we care. For the division of 10 by 3 produces a quotient of 3 with a remainder of 1, leading to a repeat of the division of 10 by 3.

Example 9.

Find the decimal expansion of 157/660.

Solution. Dividing 157 by 660 gives a quotient of 2 and a remainder of 25. Divide 25 by 660 to get a quotient of 0.03 and a remainder of 5.200. So far we have

$$\frac{157}{660} = 0.23 + \frac{5.2}{660}.$$

Continuing division of the remainder by 660 produces a quotient of 0.007 and a remainder whose numerator is 58. Now division by 660 gives a quotient of .0008 and a remainder whose numerator is 52.

Since that is what we had in the step that produced a 7, we'll again get a quotient of 8 and a remainder of 58. Furthermore, these two steps continue to repeat themselves, so we can conclude that

$$\frac{157}{660} = 0.23787878\cdots ,$$

with the sequence 78 repeating itself as often as we need. This will be written as $157/660 = 0.23\overline{78}$, where the over line indicates continued repetition.

Example 10.

Find the decimal expansion of 3/11.

Solution. Ignoring decimal points, the first division in the long division is $30 \div 11$, giving 2 with a remainder of 8. So, the second division is $80 \div 11$, giving 7 with a remainder of 3. Then the third division $30 \div 11$ is the same as the first, so we have entered the repeating cycle with 27 as the repeating number. Now take care o fthe decimal point: 3/11 is less than 1 and bigger than 0.1, so the answer is $3/11 = 0.\overline{27}$

We conclude that

> The decimal expansion of a fraction is eventually terminating or repeating - that is, after some initial sequence of digits, there a following set of digits (which may consist of zeroes) that repeats over and over.

Express Decimals as Fractions

To complete this set of ideas, we show that an eventually repeating decimal represents a fraction.

Example 11.

Let's start with: 0.33333 . . . , or in short notation $0.\overline{3}$. Let a represent this number. Multiply by 10 to get $10a = 3.\overline{3}$ We then have the two equations:

$$a = 0.\overline{3} .$$

$$10a = 3.\overline{3}$$

Substitute a for $0.\overline{3}$ in the second equation to get $10a = 3 + a$. We solve for a to get $a = 1/3$. Since $a = 0.\overline{3}$, we conclude that $0.\overline{3} = 1/3$.

Alternatively, realizing that $3.\overline{3} = 3 + 0.\overline{3}$, we could subtract the first equation from the second to get $9a = 3$, to conclude that $a = 3/9 = 1/3$ is the fraction represented by the decimal.

Here's a more complicated example: Example 12.

Convert $0.\overline{234}$ to a decimal.

Solution. . Set $a = 0.\overline{234}$. Now multiply by 1000, to get these two equations:

$$a = 0.\overline{234} .$$

$$1000a = 234 + .\overline{234}$$

The second equation becomes (after substitution) $1000a = 234 + a$, from which we conclude: $999a = 234$, so $a = 234/999$, which, in lowest terms, is $26/111$. Alternatively, we could subtract the first equation from the second, getting to $999a = 234$, and ultimately the same answer.

EXAMPLE 13.

We really should be discussing decimals that are eventually repeating, such as $0.26\overline{54}$. First we take care of the repeating part: let $b = 0.\overline{54}$, and follow the method of the preceding examples to get the equation $100b = 54 + b$, so $b = 54/99 = 6/11$ in lowest terms. Now:

$$0.26\overline{54} = \frac{26}{100} + 0.00\overline{54} = \frac{26}{100} + \frac{0.\overline{54}}{100} = \frac{26}{100} + \frac{1}{1000}\frac{6}{11} = \frac{292}{1100} = \frac{73}{275}\ .$$

For a final example, let's convert $0.2\overline{3}$. We have:

$$0.2\overline{3} = \frac{2}{10} + \frac{1}{10}\left(\frac{1}{3}\right) = \frac{7}{30}\ .$$

Expand the Number System

In the preceding, for any point on the line, we have discussed how to get a sequence of terminating decimals that provide better and better estimates (in fact, each is within one-tenth of the given point than its predecessor). If we think of a point on the line as blur on the line (as it is in reality), then we can repeat this process until our decimal expansion is inside the blur, and then indistinguishable from the point of interest. Better optical devices will shrink the blur, and so we'll have to apply our process a few more times. Now, in the idealization of the real world that is the realm of mathematics, this process may never end; and so we envision infinite decimal expansions that can be terminated at any time to give us the estimate that is as close as we can perceive with out most advanced optics. Thus the repeating decimals make sense: if we want a decimal approximation of 1/3 that is correct within one-tenthousandth, we take 0.3333. If we want a decimal approximation correct within 500 decimal points, we take 0. followed by 500 3's.

Let us review the operational description of the decimal expansion through measurement: Let a be a point on the number line. Let N be the largest integer less than or equal to a; that is $N \leq a < N + 1$. Now divide the integer between N and $N + 1$ into tenths, and let d_1 be the number of tenths between N and a. If this lands us right on a, then $a = N + d_1/10$. If not, divide the interval between $N + d_1/10$ and $N + (d_1 + 1)/10$ into hundredths and let d_2 be the number of hundredths below a, so that

$$N + \frac{d_1}{10} + \frac{d_2}{100} \leq a < N + \frac{d_1}{10} + \frac{d_2 + 1}{100}\ .$$

Now do the same thing with thousandths and continue indefinitely - or at least as far as your measuring device can take you.

Now, some decimal expansions are neither terminating, nor ultimately repeating, for example

$$0.101001000100001000001\cdots\ ,$$

where the number of 0's between 1's continues to increase by one each time, indefinitely. Another is obtained by writing down the sequence of positive integers right after each other:

$$0.123456789101112131415\cdots\ .$$

Do such expressions really define (ideal) points on the line? This question, in some form or another, has been around as long as numbers have been used to measure lengths. Today, we know that it is not a question that can be answered through common sense, or logical deduction, but, as an assertion that is fundamental to contemporary mathematics, must be accepted as a given (or, in some interpretations, as part of the definition of the line).

Such decimal expansions that are neither terminating or repeating cannot represent a rational number, so are said to be *irrational*. We can continue to make up decimal expansions that are neither terminating nor repeating, and in that way illustrate more irrational numbers. But what is important to understand is that there are constructible lengths (like the sides of some tilted squares) that are irrational.

Now, the fact that we are heading for is this: for any whole number N, if N is not a perfect square, then $\sqrt{N}$ is not expressible as a fraction; that is, $\sqrt{N}$ is an irrational number. The existence of irrational numbers was discovered by the ancient Greeks (about 5th century BCE), and they were terribly upset by the discovery, since it was a basic tenet of theirs that numbers and length measures of line segments were different representations of the same idea. What happened is that a member of the Pythagorean society showed that it is impossible to express the length of the side of the dashed square in Figure 2 ($\sqrt{2}$) by a fraction. Here we'll try to describe the modern argument that actually proves more:

> If N is a whole number then either it is a perfect square or $\sqrt{N}$ is irrational.

Another way of making this statement is this: If N is a whole number, and $\sqrt{N}$ is rational, then $\sqrt{N}$ is also a whole number. To illustrate why this is so, let's demonstrate it for $N = 2$. Suppose that $\sqrt{2}$ is a rational number. Express it as $\sqrt{2} = \frac{p}{q}$ in lowest terms (meaning that p and q have no common integral factor). In particular, it cannot be the case that p and q are both even. Now square both sides of the equation to get

$$2 = \frac{p^2}{q^2} \quad \text{so that} \quad 2q^2 = p^2 .$$

This last equation tells us that p^2 is even (it has 2 as a factor). But, since the square of an odd number is odd, that tells us that p must be even: $p = 2r$ for some integer r. Then $p^2 = 4r^2$, and putting that in the last equation above we get

$$2q^2 = 4r^2 \quad \text{so that} \quad q^2 = 2r^2 ,$$

and q is thus also even. This contradicts the statement that $2 = p/q$ in lowest terms, so we conclude that our original assumption that $\sqrt{2}$) is equal to a ratio of integers was false. Therefore, since *sqrt*2 can not be represented as a ratio of integers, *sqrt*2 is irrational.The argument for any whole number N that is not a perfect square is similar, but uses more about the structure of whole numbers (in terms of primes), It might be useful to go into this, but it is beyond the 8th grade core.

Example 14.

Show that $\sqrt{50}$ is irrational.

Solution. $50 > 7^2$ and $50 < 8^2$, so $\sqrt{50}$ lies between 7 and 8, and thus cannot be an integer. Another way to see this is to factor 50 as $25 \cdot 2$, from which we conclude that $\sqrt{50} = 5\sqrt{2}$, and $\sqrt{2}$ is irrational.

Example 15.

π is another irrational number that is is defined geometrically. How do we give a numerical value to this number? The ancient Greek mathematician, Archimedes, was able to provide a geometric way of approximating the value of π. Here we describe it briefly, using its definition as the quotient of the circumference of a circle by its diameter. In Figure 7, consider the triangles drawn to be continued around the circle 8 times.

Now measure the lengths of the diameter (5.3 cm) and the lengths of the outer edge of the triangles shown triangles. The length of the inner one is 2 cm, and of the outer one is 2.3 cm. Now, reproducing the triangles in each half quadrant, we obtain two octagons, one with perimeter 8×2, which is less than the circumference of the circle and the other with perimeter 8×2.3 which is greater than the circumference of the circle. Since we know the formula $C = \pi D$, where C is the circumference of a circle, and D its diameter, we obtain

$$8 \times 2 < C = \pi D = 5.3\pi \quad \text{and} \quad 8 \times 2.3 > C = \pi D = 5.3\pi \,.$$

After division by 5.3, these two inequalities give the estimate: $3.02 < \pi < 3.47$. To increase the accuracy, we increase the number of sides of the approximating polygon. Archimedes created an algorithm to calculate the polygon circumferences each time the number of sides is doubled. At the next step (using a polygon with 16 sides), the procedure produces the estimate 22/7 for π, which has an error less than 0.002.

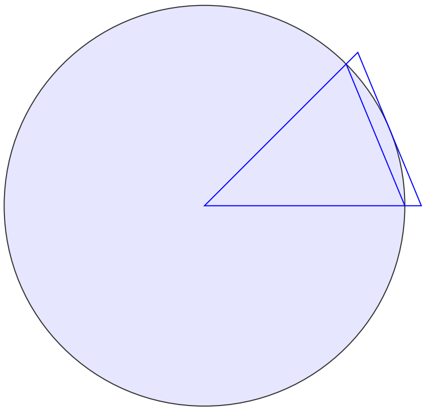


Figure 7

Approximating the Value of Irrational Numbers

Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2*). 8.NS.2*

Since irrational numbers are represented by decimals that are neither terminating nor repeating, we have to rely on the definition of the number to find fractions that are close to the irrational number - hopefully as close as we need the approximation to be. So, as we saw above, Archimedes used the definition of π to find a rational number (22/7) that is within 1/500 of π. Even better, Archimedes described an algorithm to get estimates that are closer and closer - as far as we need them to be. But how about square roots?

We know that $\sqrt{2}$ can be represented as the diagonal of a right triangle with leg lengths equal to 1, so we can measure the length of that diagonal with a ruler. But the accuracy of that measure depends upon the detail in our ruler. We'd rather have an arithmetic way to find square roots - or, to be more accurate, to approximate them. For this, we return to the discussion of decimals in 7th grade. There, a method was described that found decimals that came as close as possible to representing a given length. However, it too, depended upon our capacity to measure. What we want is a method to approximate square roots, which, if repeated over and over again, gives us an estimate of the square root that comes as close to the exact length as we need it to be.

Before calculators were available, a method for estimation of square roots was that of trial and error. With calculators, it is not so tedious - let us describe it.

EXAMPLE 16.

Approximate $\sqrt{2}$ correct to three decimal places.

SOLUTION. Since $1^2 = 1$ and $2^2 = 4$, we know that $\sqrt{2}$ is between 1 and 2. Let's try 1.5. $1.5^2 = 2.25$, so 1.5 is too big. Now $1.4^2 = 1.96$ so 1.4 is too small, but not by much: 1.4 is correct to one decimal place. Let's try 1.41: $1.41^2 = 1.9881$ and $1.42^2 = 2.0164$. It looks like 1.42^2 is a little closer, so let's try 1.416: $1.416^2 = 2.005$; so we try $1.415^2 = 2.002$, still a little large, but quite close. To see that we are within three decimal places, we check $1.414^2 = 1.9994$. That is pretty close, and less that 2. To check that 1.414 is correct to three decimal places, we check half an additional point upwards: $1.4145^2 = 2.0008$, so the exact value of $\sqrt{2}$ is between 1.4140 and 1.4145, so 1.414 is correct to three decimal places.

Example 17.

Gregory wants to put a fence around a square plot of land behind his house of 2000 sq.ft. How many linear feet of fence will he need?

Solution. His daughter, a member of this class, tells him that the length of a side will be $\sqrt{2000}$, ands since there are four sides. Gregory will need $4\sqrt{2000}$ linear feet of fence. He objects that he can't go to the lumber yard asking for $4\sqrt{2000}$ linear feet of fence; he needs a number. His daughter thinks, well that is a number, but understands that he needs a decimal approximation She says: "First we try to get close: The square of 30 is 900; so we try 40: it s square is 1600, and $50^2 = 2500$, so our answer is between 40 and 50. Let's try 45; since $45^2 = 2025$, we are close." Gregory says that for his purpose, this is close enough, but he is curious and asks his daughter if she can do better. "Sure," she says, and calculates $44^2 = 1936$, and informs her father that the answer is between 44 and 45 - and maybe closer to 45. So she calculates the squares of number between 45.5 and 50:

Number	Square of number
44.5	1980.25
44.6	1989.16
44.7	1998.09
44.8	2007.04

and concludes that each plot has to have side length between 44.7 feet and 44.8 feet, and much closer to 44.7. so she suggests that he go with 44.72; that is he will need $4 \times 44.72 = 178.88$ linear feet of fence in the meantime, his daughter checks: the square of 44.72 (accurate to two decimal points) is 1999.88 - not bad.

Example 18.

Find the square root of 187 accurate to two decimal places.

Solution. Since $10^2 = 100$, we try $11^2 = 121$, $12^2 = 144, 13^2 = 169$ and $14^2 = 196$. From this we conclude that $\sqrt{187}$ is between 13 and 14, and probably a little closer to 14. Next, try 13.7: $13.7^2 = 187.69$, a little too big. Now try 13.65: $13.65^2 = 187.005$; almost there! Now, to be sure we are correct to 2 decimal points, we calculate $13.64^2 = 186.05$. Since 13.65^2 is so much closer to 187, we conclude that 13.65 is the answer, accurate to two decimal places.

Trial and error seems to work fairly well, so long as we start with a good guess, and have a calculator at hand. But, in the field, the engineer will want a way to do this with a calculator or computer, where the computer does the guessing. Such a method is called "Newton's method," after Isaac Newton, one of the discoverers of the Calculus. Newton reasoned this way: suppose that we have an estimate a for the square root of N, That is: $a^2 \sim N$, where the symbol $\sim$ means "is close to." Dividing by a, we have $a \sim N/a$, so N/a is another estimate, about as good as a. He also noticed that a and N/a lie on opposite sides of $\sqrt{N}$: as follows: suppose $a < \sqrt{N}$, so that $a^2 < N$. Multiply both sides by N, giving $Na^2 < N^2$. Now divide both sides by a^2 to get:

$$N < \frac{N^2}{a^2} \quad \text{or} \quad \sqrt{N} < \frac{N}{a} \ .$$

Because of this, the average of a and N/a average should be an even better estimate . So he set

$$a' = \frac{1}{2}\left(a + \frac{N}{a}\right) ,$$

and then repeated the logic with a', and once again with the new estimate, until the operation of taking a new average produced the same answer, up to the desired number of decimal places. The amazing fact is that Newton showed that this actually works and in a small number of steps, and can be applied to find solutions of a wide range of numeric problems. Let's now illustrate inf the following example.

Example 19.

Find the square root of two accurate up to four decimal places.

Solution. We try 1 and 2, getting $1^2 = 1$ and $2^2 = 4$. Now, $2/1 = 2$, so according to Newton, we should now try the average: 1.5. Since $2/(1.5) = 4/3$, we next try the new average:

$$a' = \frac{1}{2}\left(1.5 + \frac{4}{3}\right) = \frac{17}{12} - 1.41667\ .$$

This already is a pretty good estimate. Now, let us bring in calculators or a computational program like Excel so that we can repeat this process until the new estimate is the same as the second. These Excel calculations (using Newton's method) are shown in the first table of Figure 8. Our discussion brought us to the second line (1.416666667 is 17/6 correct to 9 decimal places). The table continues with Newton's method, until the estimate (last column) stabilizes. Actually, it stabilized (to four decimal places) in the third line, but we have continued the calculation, to show that there is no further change. In fact each step in Newton's method produces a more accurate estimate, so once we have no change (to the number of correct decimals we require), there is no need to go further.

Example 20.

Use Newton's method to find the square root of 5.

Solution. First of all "find the square root of five" is not very meaningful. Using the tilted square with side lengths 1 and 2, we found the square root of 5 as a length. So, perhaps here wen mean, "find the numerical value of the square root of 5." But as we have already observed, $\sqrt{5}$ is not expressible as a fraction, so we can't expect to "find" its value precisely. What we can hope for is to find a decimal expansion that comes as close as we please to $\sqrt{5}$. So, let us make the question precise: find a decimal that is an estimate of $\sqrt{5}$ that is correct to 4 decimal places. Let's go through Newton's method.

First we see that 2 is a good approximation for $\sqrt{5}$ by an integer, since $2^2 = 4$ and $3^2 = 9$.Now $5/2 = 2.5$ is also a good guess, since $2.5^2 = 6.25$. In fact it is what Newton's method wants us to look at next. The rest of the computation was done by Excel and is shown in Figure 8. There we see that by the third step in Newton's method, we have $\sqrt{5}$ correct to 6 decimal places.

Do we ever get to the place where the "last two approximations" agree perfectly - that is: when have we arrived at the numerical value of the desired point? Alas, for irrational numbers, the answer is "we never do, but we do get better and better." Unless N itself is a perfect square (the square of another integer), we will never arrive at a decimal expansion that is precisely $\sqrt{N}$. But - and this is all we really need - we can get as close as we want.

Example 21.

Now use Newton's method to calculate $\sqrt{25}$ and $\sqrt{1000}$.

Solution. The calculation is exhibited in the third and fourth tables of Figure 8. We could also note that $\sqrt{1000} = 10\sqrt{10} = 10\sqrt{2}\sqrt{5}$ and multiply the results of the first two calculations.

Example 22.

It is not necessary that the first guess at the square root is close to the answer - we can start with any positive number and end up with the estimate we want - it just may take a little longer. To see this, use Newton's method to find $\sqrt{150,000}$, starting with 2 as the first guess. See Figure 9.

We can extend arithmetic relations and operations to irrational numbers since the decimal expansion allows us to

Figure 8

Square Root of 2 (Example 19)

First estimate: $a = 1$

Estimate	$2/a$	New Estimate (Average)
1	2	1.5
1.5	1.333333	1.416666667
1.41667	1.411765	1.414215686
1.41422	1.414211	1.414213562
1.41421	1.414214	1.414213562
1.41421	1.414214	1.414213562
1.41421	1.414214	1.414213562
1.41421	1.414214	1.414213562

Square Root of 5 (Example 20)

First estimate: $a = 2$

Estimate	$5/a$	New Estimate (Average)
2	2.5	2.25
2.25	2.222222	2.236111111
2.23611	2.236025	2.236067978
2.23607	2.236068	2.236067977
2.23607	2.236068	2.236067977
2.23607	2.236068	2.236067977
2.23607	2.236068	2.236067977
2.23607	2.236068	2.236067977

Square Root of 25 (Example 21)

First estimate: $a = 1$

Estimate	$25/a$	New Estimate (Average)
1	25	13
13	1.923077	7.461538462
7.46154	3.350515	5.406026963
5.40603	4.624468	5.015247602
5.01525	4.984799	5.000023178
5.00002	4.999977	5
5	5	5
5	5	5

Square Root of 1000 (Example 21)

First estimate: $a = 25$

Estimate	$1000/a$	New Estimate (Average)
25	40	32.5
32.5	30.76923	31.63461538
31.6346	31.61094	31.62277882
31.6228	31.62277	31.6227766
31.6228	31.62278	31.6227766
31.6228	31.62278	31.6227766
31.6228	31.62278	31.6227766
31.6228	31.62278	31.6227766

Square root of 150000 (Example 22)

First estimate:		$a = 1$
Estimate	$150000/a$	New Estimate (Average)
1	150000	75000.5
75000.5	1.999987	37501.24999
37501.2	3.999867	18752.62493
18752.6	7.99888	9380.311905
9380.31	15.99094	4698.151422
4698.15	31.92745	2365.039437
2365.04	63.42389	1214.231663
1214.23	123.5349	668.8832869
668.883	224.2544	446.5688284
446.569	335.8945	391.2316493
391.232	383.4046	387.3181067
387.318	387.2786	387.2983351
387.298	387.2983	387.2983346
387.298	387.2983	387.2983346
387.298	387.2983	387.2983346

Figure 9

get as close as we please to any number.

Example 23.

What is bigger π^2 or 10?

Solution. To answer such a question, we need to find a rational number larger than π whose square is smaller than 10. 3.15 will do, since it is larger than π and $3.15^2 = 9.9225 < 10$.

But we do have to be a little careful: the closeness of approximations changes as we add or multiply them. So, if a agrees with a_0 up to two decimal points, and b agrees with b_0 up to two decimal points, we cannot conclude that $a + b$ or ab agree with $a_0 + b_0$ or $a_0 b_0$ up to two decimal places. Let us illustrate that.

Example 24.

3.16 agrees with $\sqrt{10}$ to two decimal places. But $3.16 \times 3.16 = 9.9856$, which does not agree with 10 to two decimal places.

Example 25.

Approximate $\pi + \sqrt{2}$ to three decimal places. Start with the approximations up to four decimal points: 3.1416 and 1.4142, and add: $3.1416+1.4142 = 4.5558$, from which we accept 4.556 as the three (not four) decimal approximation because of the rounding to the fourth place in the original approximations.

For products it is not so easy. Suppose that A and B are approximations to two particular numbers, which we can denote as A_0, B_0, Then we have $A_0 = A + e$, $B_0 = B + e'$ where e and e' are the errors in approximation. For the product we will have $A_0 B_0 = (A + e)(B + e') = AB + Ae' + Be + ee'$, telling us that the magnitude of the error has been multiplied by the factors A and B, which may put us very far from the desired degree of accuracy.

Example 26.

Suppose that A and B approximate the numbers 1 and 100 respectively within one decimal point. That means that

$$0.95 < A < 1.05 \quad \text{and} \quad 99.95 < B < 100.05 .$$

When we multiply, we find thatAB could anywhere in the region

$$94.9525 < AB < 105.0525 \,,$$

thus, as much as 5 units away from the accurate product 100.

To find approximate values for irrational numbers, we have to understand the definition of the number so we can use it for this purpose. For example, suppose we want to find a number whose cube is 35, correct up to two decimal places. Start with a good guess. Since $3^3 = 27$ and $4^3 = 64$, we take 3 as our first guess. Since 27 is much closer to 35 than 64, we now calculate the cubes of 3.1, 3.2, 3.3, ... until we find those closest to 35: $3.1^3 = 29.791$, $3.2^3 = 32.6768$, $3.3^3 = 35.937$. We can stop here, since the last calculated number is larger than 35. Now we go to the next decimal place, starting at 3.3, working down (since 35.937 is closer to 35 than 32.6768: $3.29^3 = 35.611$, $3.28^3 = 35.287$, $3.27^3 = 34.965$. Since the last number is as close as we can get to 35 with two place decimals, we conclude that 3.27 is correct to two decimal places.

Chapter 8
Integer Exponents, Scientific Notation and Volume

We have already introduced the notation x^2 for $x \times x$ and x^3 for $x \times x \times x$, and it is easy to see how to extend this to all positive integers: x^n just means that we multiply x by itself n times. So, $2^6 = 64$, $3^4 = 81$ and so forth. Since we are using whole numbers just to count factors, clearly $x^{m+n} = x^m \times x^n$: multiplying x to itself m times, and then n more times is the same as multiplying x by itself $m + n$ times. We then ask the question: can we make sense of x^p for *all* integers p, so that the usual rules of arithmetic on the exponents apply? The answer is "yes," and the exploration of this is the content of the first section of this chapter. We put particular emphasis on the assertion that $x^0 = 1$ for all numbers $x \neq 0$. There are several ways to to see that this is the right definition, but the fact is that it is simply a consequence of the rules of arithmetic, as we shall show.

In the next section we revisit place value, recalling that when a number is exhibited in place 10 notation, each place represents a power of ten, we move on to a shorthand for representing numbers, using exponential notation. Scientific notation is important, not just as a convenience for dealing with very small or very large numbers, but as a way of understanding "orders of magnitude." When it is said that phenomenon A is two orders of magnitude more likely than phenomenon B (as in the scale for Hurricane intensity) we do not mean that A is twice as likely as B; we mean that A is 100(= 102) times more likely than B. We pose many problems illustrating the meaning of "orders of magnitude" that should convince students that this is not just a shorthand, but conveys a rich meaning that other wise could be missed.

Finally, in the last section we introduce certain volume calculations (for a cylinder, cone and sphere), the purpose of which is to work with the relations among these solid figures, and secondly, to apply the mathematics of the preceding sections. As an example, Farmer Brown has fields that can produce grain, and silos that can store them. Given the correspondence between square feet of farmland and cubic feet of grain, we ask these questions: a) for a certain size of field, what storage capacity is needed. Given the size of the silo, how many square feet need to be planted in wheat so as to fill the silo?

Section 8.1 Integer exponents

Know and apply the properties of integer exponents to generate equivalent numerical expressions. For example, $3^2 \times 3^{(-5)} = 3^{(-3)} = 1/(3^3) = 1/27$. *8.EE.1*

In previous discussions about area and volume we already have introduced the notation x^2 and x^3: the first is the product of two x's, the second the product of three x's. We can now introduce the same notation for all counting numbers (positive integers): x^n is the product of n x's for any positive integer n. Notice that multiplication of such objects amounts to addition in the exponent: $x^3 \times x^8 = x^{(3+8)} = x^{11}$. In general, we can say that for positive integers, we have

$$x^p \times x^q = x^{p+q}$$

for all positive integers p and q.

Can we extend this notation to all integers, positive, negative and zero? If we want to be able to write

$$x^5 \times x^{-3} = x^{5-3} = x^2 ,$$

we have to understand multiplication by by x^{-3} as an operation going from x^5 to x^2. But we know such an operation: it is that of canceling three of the x's, and this is the same as division by x^3. So, we take this as the definition of negative exponents:

$$x^{-p} = \frac{1}{x^p}$$

The addition rule above now holds for all nonzero integers.

Since x^2 means: multiply the expression x by itself, if we replace x by x^3, then $(x^3)^2 = x^3 \times x^3 = x^6$. But x^6 is the same as $x^{2\cdot 3}$. This of course is true for all integers, not just 2 and 3, so we have this understanding of the product rule for exponents:

$$x^{p\cdot q} = (x^p)^q = (x^q)^p, \quad \text{for all positive integers } p \text{ and } q.$$

Now what meaning do we attach to the expression x^0? We follow the logic of the rules of arithmetic: If we start with the expression x^5, and cancel all of the x's, we get 1. That is, we have:

$$x^5 \times x^{-5} = 1 .$$

The rules of arithmetic tell us that $x^{5+(-5)} = x^0$, so we need to adopt the rule $x^0 = 1$.

Another way to look at it is this: we are studying the multiplicative structure of expressions. When we work with the additive structure of expressions, the simplest expression is 0. So, in the case of multiplicative structure, the simplest expression should be 1. It follows that anything multiplied by itself no times is 1. Finally, we have to understand that these rules apply to all values for x except 0, because the logic doesn't apply for $x = 0$.. We can't divide by zero, so 0 to a negative exponent doesn't make sense.

Let's pay particular attention to raising negative number to a power. Since $-a$ and $(-1)(a)$ are the same thing, we can calculate using the commutative property. For simplicity, suppose $a > 0$, so we already feel comfortable with a^n. How about $(-a)^n$? Write this as$(-1)^n \times a^n$, and so we just have to know what $(-1)^n$.If n is a positive integer, this is the product of n (-1)'s. Now multiplication by -1 is the same as reflection in the origin, so $(-1)^2$ is the reflection of -1 in the origin, and is 1; $(-1)^3)$ reflects -1 in the origin and then reflects back again, so is -1. Continuing with this representation as reflecting back and forth around the origin, we see that $(-1)^n$ is 1 if n is even, and is -1 if n is odd. This is true also for negative exponents, since $(-1)^{-1} = -1$ (-1 is its own multiplicative inverse).

Let us summarize the operational techniques with exponents:

In the following, a and b can be any number, and p , q and n are integers (positive or negative):

- $a^{p+q} = a^p \times a^q$
- $(\frac{a}{b})^n = a^n b^{-n}$
- $(a \cdot b)^n = a^n \times b^n$
- $(a^p)^n = a^{p+n}$
- $a^{-p} = \frac{1}{a^p}$.
- $(-a)^n$ is equal to a^n if n is even, and is equal to $-(a^n)$ if n is odd.

It is good to have clearly in mind that the use of exponents is in reference tot the multiplication of positive numbers, and that, for $a > 1$, the numbers a^n, for $n > 0$, are to right of 1, and for $n < 0$ between 0 and 1. Figure 1 illustrates this for $a = 2$.

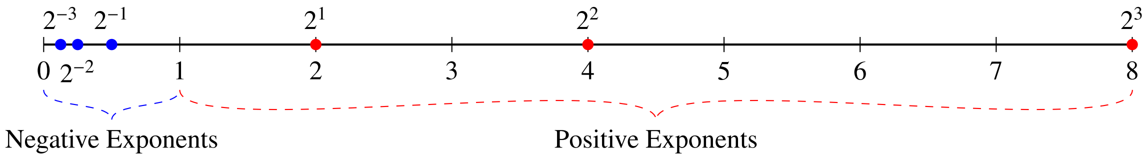


Figure 1.

It is good to have as a reference the values of the powers of small digits.

n^1	n^2	n^3	n^4	n^5	n^6	n^7	n^8	n^9	n^{10}
1	1	1	1	1	1	1	1	1	1
2	4	8	16	32	64	128	256	512	1028
3	9	27	81	243	729	$\cdots$			
4	16	64	256	1028	$\cdots$				
5	25	125	625	$\cdots$					
10	100	1000	10000	$\cdots$					

The powers of 10 are particularly easy: for p positive, 10^p is a 1 followed by p zeros, and 10^{-p} is a decimal point followed by zeros and ending in the pth position with a 1:

$$10^2 = 100 \quad 10^5 = 100,000 \qquad 10^{-2} = .01 \quad 10^{-5} = .00001 .$$

One of the values of exponents is that their use makes the understanding of our *place value* notation and computation more clear. The number 5283.7 is expressed in "long form" as

$$5 \times 1000 + 2 \times 100 + 8 \times 10 + 3 + 7 \times \frac{1}{10} .$$

Using exponents this becomes

$$5 \times 10^3 + 2 \times 10^2 + 8 \times 10^1 + 3 \times 10^0 + 7 \times 10^{-1} .$$

Example 1. Place Value Operations

- Double 63. To double a number we could multiply the number by 2. Or, we could double every digit. This way, double 63 is 126. But what if we double 67, is the answer 1214? No, because the "place" value of the 2 in 12 and the one in 14 are the same and so must be added, and the answer is 134. Although this looks like a magical trick - it is not. Writing out the place value in long form, doubling the digits looks like this:

  $$2\times67 = 2\times(6\times10+7) = 2\times(6\times10)+2\times7 = 12\times10+14 = 12\times10+1\times10+4 = 13\times10+4 = 134$$

- Multiply 102 by 54. We put these numbers in long form and use the rules of arithmetic:

  $$102 \cdot 54 = (10^2 + 2)(5 \times 10 + 4) = 5 \times 10^3 + 4 \times 10^2 + (2 \cdot 5) \times 10 + 2 \cdot 4$$
  $$= 5 \times 10^3 + 4 \times 10^2 + 10^2 + 8 = 5000 + 500 + 8 = 5508 .$$

- Take 3 percent of 5000. In exponential notation, 3 percent is 3×10^{-2} and 5000 is 5×10^3. So, we use the rules of exponents to solve:

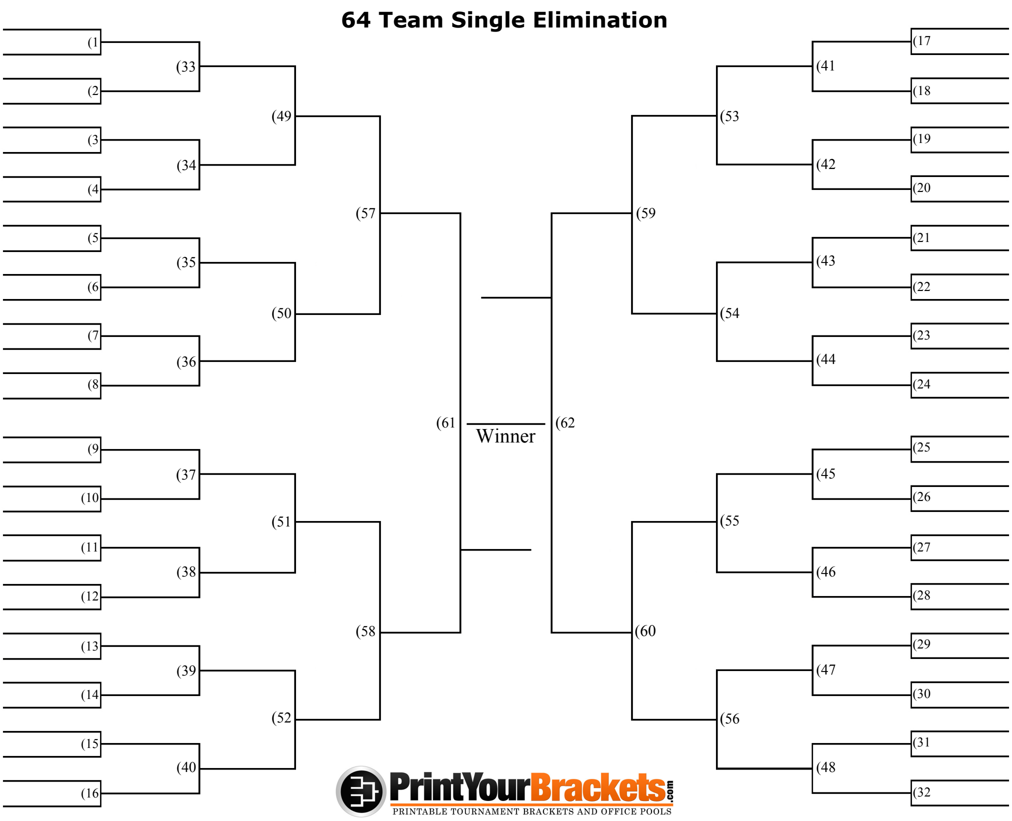
  $$(3 \times 10^{-2}) \cdot (5 \times 10^3) = 15 \times 10^{-2+3} = 15 \times 10^1 = 150 .$$

Example 2. Operations with exponents

- $8 \cdot 16 = 2^3 \cdot 2^4 = 2^7 = 128$.
- $64 \cdot 18 = 2^6 \cdot 2 \cdot 3^2 = 2^7 \cdot 3^2 = 128 \cdot 9 = 128 \cdot (10 - 1) = 1280 - 128 = 1152$.
- Approximate $\sqrt{6340}$. This is a little less that 64×100, so its square root is a little less than $8 \times 10 = 80$.

Example 3. March Madness (from the Chicago Maroon, March, 2012)

March Madness - the NCAA final basketball tournament - has the form of a *single-elimination* tournament. In such a tournament, we start with a certain number of teams, and we pair them off into games: each team plays a game. This is called the *first round*. All the losers in the first round are eliminated; in the second round all the winning teams are paired off into games, and all the second round losers are eliminated. This process continues until only two teams remain: this is the *final round* and the winner is the champion of the tournament. For a graphic of the 2012 women's basketball *brackets* see the figure below.



Since there are two teams in the final round, there had to be four teams in the semi-final round, and thus eight teams in the preceding round and so forth. So, it is necessary, for a single elimination tournament to work, with no teams ever idle, that we start with a number of teams that is a power of two, and that exponent is the number of rounds. So, for example, if we start with 16 teams, since 16 = 24, there are 4 rounds and $8 + 4 + 2 + 1 = 15$ games.

In March Madness we start with 64 teams. How many rounds are there? How many teams are in the second round? in any round? How many games total are played?

Solution. Since $64 = 2^6$, there are six rounds. Each round eliminates half the remaining teams, so there are 32 teams in the second round, 16 in the third, and so forth. There are $32 + 15 + 8 + 4 + 2 + 1 = 63$

games. Another way of counting is that there are 63 teams that are NOT champions, and each game produces one non-champion.

Section 8.2 Scientific Notation

Scientific Notation and Place Value

Use numbers expressed in the form of a single digit times an integer power of 10 to estimate very large or very small quantities, and to express how many times as much one is than the other. For example, estimate the population of the United States as 3×10^8 *and the population of the world as* 7×10^9*, and determine that the world population is more than 20 times larger. 8.EE.3*

In today's world we work with very big numbers (astronomical distances) and very small numbers (microscopic distances), so a shorthand has been invented to make it easier to handle such numbers. To illustrate, suppose we want the product of 300,000,000 and 7000. We know that this is going to be 21 followed by a certain number of zeros, but how many? We calculate this way:

$$300,000,000 \times 7000 = (3 \times 100,000,000)(7 \times 1000) = (3 \times 7)(100,000,000 \times 1000)$$

$$= 21 \times 100,000,000,000 = 2,100,000,000,000 \ .$$

The second multiplication $(100,000,000 \times 1000)$ amounts to putting the 0's at the end of the second factor behind the zeros of the first factor; in other words, the number of zeros in the product is the sum of the numbers of zeros in the factors. In exponential notation: $!0^8 \times 10^3 = 10^{11}$. Using this notation, the above calculation now looks like this:

$$(3 \times 10^8)(7 \times 10^3) = 21 \times 10^{11} \ .$$

Whenever we write numbers using the symbol $\times 10^n$, we say are using *scientific notation*. As this example shows, this notation makes calculations easier to read. It also makes it easier to make comparisons; for example the statement "Earth is 93 million miles away from the Sun, and Mars is 143 million miles away from the Sun," we are using scientific notation (simply replace the word "million" with $\times 10^6$. This is easier to understand than the statement "Earth is 93000000 miles away from the Sun, and Mars is 143000000 miles away from the Sun."

To be precise, a number is said to be written in *normalized scientific notation* if it is given in the form $a \times 10^n$, where a is a number whose absolute value is greater than or equal to 1 and strictly less than 10 and n is an integer. a is called the *significant figure* of the number, and n its *order of magnitude*. To illustrate: Earth is 9.3×10^7 miles from the Sun, and Mars is 1.43×10^8 miles from the Sun and $(3 \times 10^8)(7 \times 10^3) = 2.1 \times 10^{12}$.

Example 4.

- Express 35,000,000 in normalized scientific notation. This is 35 followed by 6 zeros, so is 35×10^6. To put this in normalized scientific notation, we have to move the decimal point one place to the left, and raise the exponent by 1, giving us the answer: 3.5×10^7. Note that we could also write 35000000 as 350×10^5 or 0.35, depending upon what it is that we want to emphasize. Generally speaking, as every three places is denoted by a comma, it is best to go by multiples of 3.
- Express 3,650,000 in scientific notation. This could be 365×10^4 or 3.65×10^6; it is the second that is in normalized scientific notation.
- Express 3,651,284 in scientific notation with 3 significant figures. This means that, for purposes of estimation, we care only about the first three digits, and so the answer is 3.65×10^6.

Computers and calculators use a different notation for scientific notation: 36400000 appears as $3.64E7$, meaning 3.64×10^7. Let us take a moment to introduce the vocabulary for large numbers.

Name	Number	Scientific Notation
One	1	10^0
Ten	10	10^1
Hundred	100	10^2
Thousand	1000	10^3
Ten Thousand	10,000	10^4
Hundred Thousand	100,000	10^5
Million	1,000,000	10^6
Billion	1,000,000,000	10^9
Trillion	1,000,000,000,000	10^{12}
Quadrillion	1,000,000,000,000,000	10^{15}

Table 1

and so forth. Notice one value of scientific notation: the middle column grows more and more unreadable, while the last column can be grasped. So, for example, a septillion will be 10^{24}, or 1 followed by 24 boring zeros.

Fractional decimals can be similarly written, using scientific notation, but this time the powers of 10 are negative:

$$0.1 = 10^{-1} \quad 0.001 = 10^{-3} \quad 0.367 = 3.67 \times 10^{-1} \; .$$

As far as names go, we speak of powers of 10 in the denominator by adding "th" to the end of the word, so 0.3 is 3 tenths, or 3×10^{-1}, 0.000005 is 5 millionths, or 5×10^{-6}, and so forth. You might want to notice that there is a little anomaly in scientific notation: $10,000 = 1 \times 10^4$ but $.00001 = 1 \times 10^{-5}$; that is, the exponent of 10 is not always the number of zeros from the decimal point. This is because the first place to the left is the 0th place, while the first place to the right is the (-1)st place. Finally, there do not exist special names for the negative powers of ten, but there are such names in the metric system (for grams, meters, etc.), as in this table:

Name	Number	Scientific Notation
meter	1 meter	10^0 meter
decameter	tenth of meter	10^{-1} meter
centimeter	hundredth meter	10^{-2} meter
millimeter	thousandth meter	10^{-3} meter
micrometer	millionth of a meter	10^{-6} meter
nanometer	billionth of a meter	10^{-9} meter
angstrom	tenth of a nanometer	10^{-10} meter

Table 2

An *angstrom* is the unit of measurement used to measure lengths at the atomic level.

It now makes sense to inquire: how do we calculate arithmetic operations in scientific notation? First, as for addition, the issue shouldn't come up: if the numbers are not of the same order of magnitude, the question won't come up. So, for example, $3.7 \times 10^4 + 6.1 \times 10^4 = 9.8 \times 10^4$, by the distributive property of arithmetic. But we won't be asked to calculate

$$7.104 \times 10^7 + 2.100 \times 10^2 = ?$$

because it just doesn't make sense: the first term is 5 orders of magnitude large than the second. It is like asking: if I weigh myself on a scale , and then directly after a fly lands on my head, will the scale show a difference? The answer is clearly "No." So, in the displayed question, the first number has 3 decimal points of accuracy (so is 7104×10^4), and thus the second number, 2.100×10^2, is under the radar.

However, orders of magnitude in scientific notation play a major role when the problem involves multiplication and division, For example, 120 million divided by 30 gives 4 million. The fact that "120 million" and "30" have different order of magnitude is very relevant to answering the question. Writing "120 million" as 1.2×10^8 and "30" as 3×10, this calculation becomes:

$$\frac{1.2 \times 10^8}{3 \times 10} = \frac{1.2}{3} \times \frac{10^8}{10} = 0.4 \times 10^7 = 4 \times 10^6 = 4,000,000 .$$

We can summarize this as follows:

- $(a \times 10^n)(b \times 10^m) = (a \times b)(10^{n+m})$
- $\frac{(a \times 10^n)}{(b \times 10^m)} = \frac{a}{b} \times (10^{n-m}$

Example 5.

a. Multiply 3.2×10^4 by 6×10^{-1}.
b. Divide 3.3×10^6 by 1.1×10^5.

Solution.

a. First, let's remind ourselves what we are being asked without scientific notation: Multiply 32,000 by 0.6. Since that is what is being asked, let's go back to scientific notation:

$$(3.2 \times 10^4) \times 6 \times 10^{-1} = ((3.2) \times 6) \times (10^4 \times 10^{-1} = 19.2 \times 10^3 = 1.92 \times 10^4$$

which is, in standard notation 19,200.

b. Here we are asked to divide 3.3 million into 110,000 parts.

$$\frac{3.3 \times 10^6}{1.1 \times 10^5} = \frac{3.3}{1.1} \times \frac{10^6}{10^5} = 3 \times 10$$

or 30.

Example 6.

Suppose that we want to multiply 3 billionths by 7 ten-thousandths. We might write:

$$\frac{3}{1000000000} \times \frac{7}{10000} = \frac{3 \cdot 7}{1000000000 \cdot 10000} = \frac{21}{10000000000} ,$$

but the following is much easier to understand:

$$(3 \times 10^{-6})(7 \times 10^{-4}) = 21 \times 10^{-10} .$$

Solve Problems and Apply Scientific Notation

Perform operations with numbers expressed in scientific notation, including problems where both decimal and scientific notation are used. Use scientific notation and choose units of appropriate size for measurements of very large or very small quantities (e.g., use millimeters per year for seafloor spreading). Interpret scientific notation that has been generated by technology.8.EE.4

Example 7.

- How many millions are there in a trillion? We write a million as 10^6 and a trillion as 10^{12}, and the question is: evaluate $\frac{10^{12}}{10^6}$. The answer is $10^{12-6} = 10^6$, or a million. A trillion is a million million.
- $.0031 \times 562.1$ =? The easiest way to get the answer is to use a calculator. However, we may just want an estimate, in which case, moving to scientific notation is best. Rewrite the problem as $(3.1 \times 10^{-3})(5.621 \times 10^2)$. Now estimate the significant figures: this is about 3 times 5.5, which is 16.5. Next, add the exponents, and write the (estimate of) the answer as 16.5×10^{-1}, or 1.65. The calculator turns up the accurate answer: 1.7425.
- About how much is 40% of 140 million? Rewrite this as the product

  $$(40 \times 10^{-2})(140 \times 10^6) = (40 \times 140)(10^{-2} \times 10^6) = (4 \times 14)(10^2 \times 10^{-2} \times 10^6) = 56 \times 10^6$$

  or 56 million.

Here are some illustrations of the value of scientific notation in applications, particularly to problems that give meaning to the concept "order of magnitude."

Example 8.

In a class action suit, 4000 claimants were offered a $800 million settlement. How much is that per claimant?

In scientific notation, the question is to evaluate $(8 \times 10^8) \div (4 \times 10^3)$, which is $(8 \div 4)(10^{(8-3)})$ which simplifies to 2×10^5. Thus each claimant would receive $200,000.

Example 9.

We read in the paper that the United States has a 15 trillion dollar debt. Let's say that there are 300 million working people in the United States. How much is the debt per worker?

In scientific notation this is 15×10^{12} divided by 3×10^8, which is 5×10^4, or about $50,000 for each tax-paying citizen.

Example 10.

Tameka has a job at which she earns $10 hour. Her tax rate is 18%. Let's assume that *all* of Tameka's taxes go toward paying off the $50,000 debt of the preceding problem. How many hours will she have to work to pay off her share of the debt? If she works 2×10^3 hours a year, how many years is that?

Solution. Given the assumptions of this problem, Tameka's hourly contribution to paying the debt is 18% of $10, or $1.80. Let$h$ represent the number of hours it takes until she pays off her $50,000. This gives us the equation $1.8h = 5 \times 10^4$, and thus $h = (5/1.8) \times 10^4$, which is 27,778. If she works 2000 hours in a year, that comes to $27.78 \div 2 = 13.89$ years. (Track the implicit use of scientific notation).

The following two examples are taken from Grade 7, but are repeated here to demonstrate the value of scientific notation.

Example 11.

The National Press Building on Fourteenth Street and Avenue F is 14 stories high, with 12 feet to each story. It has 150 feet of frontage on 14th St, and 200 feet on Ave F. The building has the shape of a rectangular prism. What is its volume?

We view the building as constructed by drawing upwards a 150×200 rectangle for 14 stories. Now the area of the base is $1.5 \times 10^2 \times 2 \times 10^2 = 3 \times 10^4$ sq. ft. Since each story is 12 feet high, the volume of each story is $12 \times 3 \times 10^4 = 3.6 \times 10^5$ cu. ft., and as the building is made up of a stack of 14 stories

Figure 2

identical to the first one, the total volume is $14 \times 3.6 \times 10^5 = 5.04 \times 10^6$ cu. ft; approximately 5 million cubic feet.

As it turns out, the building sold in 2011 for $167.5 million dollars, which comes to about a $33.5 cost per cu. ft. However, the value of a building is not measured by its volume, but by the square footage of its floors. Since there are 14 floors , each a copy of the first floor, so each of 30,000 square feet, the total floor area of the building is $14 \times 3 \times 10^4 = 4.2 \times 10^5$ square feet, and the cost per square foot is $(167.5 \times 10^6) \div (4.2 \times 10^5) = 39.88 \times 10$, that is $ 398.80 per square foot.

Example 12.

The Pentagon, the headquarters of the U.S. Department of Defense is a regular five-sided figure with 6.5 million square feet of floor space on seven levels, two of which are underground. The side length of the interior central plaza is about 1/4 the side length of the building.

Figure 3

a. What is the *footprint* of the Pentagon? The footprint is the total area occupied by the building together with the central plaza.

b. What is the area of the central plaza?

c. There are 11 feet of elevation between floors of the Pentagon. What is the total volume of the above-ground building?

Solution.

a. The image shows the Pentagon to be a prism - in the sense that all floors are of the same shape and size indeed all sections by planes parallel to the ground are of the same shape and size. Thus each floor of the building comprises 1/7 of 6.5×10^6 sq. ft, or 928×10^3 sq. ft. But this is the area of the base floor of the building, not the footprint, which includes the central plaza, We are told that the the length of a side of the plaza is one-third the side length of the building. Since the plaza and the building have the the same shape, that tells us that the footprint of the plaza is a downscaling of the footprint of the entire Pentagon by a linear scale factor of 1/3. Since area scales by the square of the linear scale factor, we conclude that the area of the plaza is 1/9th of the are of the footprint. Thus the area of the floor of the building, 928,000 square feet is 8/9 of the area of the footprint. The answer then, to a) is that the area of the footprint is $\frac{9}{8}(928 \times 10^3) = 1.044 \times 10^6$ sq. ft.

b. The plaza is 1/9 of the footprint, so its area is $\frac{1}{9}(1.044 \times 10^6) = 116 \times 10^3$ sq.ft.

c. The reason this figure (the volume of the building) is interesting is to estimate the cost of heating the building in winter, and air-conditioning it in summer. So, now we are interested only in the volume of the building that is above ground. Since there are 5 stories above ground, each of height 11 feet, the building stands 55 feet high. The area of the base is 928,000 sq. ft., so the volume of the building above ground is $55 \times 928,000 = 51,040,000$ cu. ft.

Now, suppose that the cost of a building in the Washington DC area is about $398.80 per sq. ft. So, at 6.5 million square feet, the cost of the Pentagon today would be approximately $4 \times 10^2 \times 6.5 \times 10^6 = 26 \times 10^8$, or 2.6 billion dollars.

Example 13.

On the computer a *byte* is a unit of information. A typical document contains many tens of thousands of bytes, and so it is customary to use these words: 1 kilobyte = 1000 bytes; 1 megabyte = 1000 kilobytes, 1 gigabyte = 1000 megabytes, 1 terabyte = 1000 gigabytes.

a. Rewrite this vocabulary in scientific notation. How many bytes are there in each of these terms?

1 kb = 10^3 b; 1 mb = 10^3 kb = 10^6 b; 1 gb = 10^3 mb = 10^9 b; 1 tera = 10^3 gb =10^{12} b.

b. My computer has a memory (storage capacity) of 16 gigabytes. How many such computers do I need to have, when all are combined, a terabyte of memory?

The question is: how many times does 16 gb go into 10^3 gb?

$$\frac{10^3}{16} = \frac{100}{16} \times 10 = 6.25 \times 10 = 62.5$$

so I'll need 63 such computers.

c. An online novel consists of about 250 megabytes. How many novels can I store on my 16 gigabyte computer?

16 gb is 16×10^3 mb, so we want to know how many times 250 goes into 16×10^3. Since we will be dividing by 250, it is worthwhile noting that

$$\frac{1}{250} = \frac{4}{1000} = 4 \times 10^{-3} .$$

Thus

$$\frac{16 \times 10^3}{250} = 16 \times 10^3 \times 4 \times 10^{-3} = 64 .$$

Example 14.

Many chemical and physical phenomena happen in extremely small periods of time. For that reason, the following vocabulary is used: 1 second = 1000 milliseconds, 1 millisecond = 1000 microseconds, 1 microsecond = 1000 nanoseconds.

a. Rewrite this vocabulary in scientific notation. How many nanoseconds are in a millisecond? in a second? in an hour?

b. My computer can download a byte of information in a millisecond. How long will it take to download a typical book (250 megabytes)? How long will it take to download the Library of Congress (containing 36 million books). Express your answer conveniently in terms of time.

Solution.

a. 1 second = 10^3 milliseconds = 10^6 microseconds = 10^9 nanoseconds. Otherwise put, a nanosecond is one billionth of a second, a microsecond is one millionth of a second, and a millisecond is one thousandth of a second.

b. It takes 10^{-3} seconds to download one byte of information, so the rate is 10^{-3} seconds per byte. Thus to download 250 megabytes, which is 2.5×10^5 bytes, so it takes $2.5 \times 10^5 \times 10^{-3} = 2.5 \times 10^2 = 250$ seconds, or 4 minutes and 10 seconds. To download 35×10^6 such books will take $36 \times 10^6 \times 2.5 \times 10^2 = 9 \times 10^9$ seconds. Since there are 3600 seconds in an hour and 24 hours in a day, in days this is:

$$\frac{9 \times 10^9}{36 \times 10^2 \times 24} = \frac{25 \times 10^7}{24} = 1.04 \times 10^7$$

days; far too long to wait. My computer is too slow. If I get a new computer that is one million times as fast (one millionth is 10^{-6}), it will still take 10.4 days to download the library of Congress.

Section 8.3 Volume

Know the formulas for the volumes of cones, cylinders and spheres and use them to solve real-world and mathematical problems. 8.G.9

In this section we start by reviewing some of the terminology and ideas of 7th grade used in volume computations. There we talked of prisms and cones based on polygonal figures in the plane; here we move to the same concepts, based on circular figures.

Prisms and Cylinders

We conceive of measurements in the various dimensions (length, area, volume) as a natural progression through the dimensions. So,to begin: we start with a point on the line that we draw out along the line for a certain distance, creating a line segment, and the measurement of that line segment is its *length*. Now suppose that we start with a line segment in the plane of length l and draw it out for a distance w perpendicular to the line segment: we obtain a rectangle of side lengths l and w.The *area* of that rectangle is the product: $A = l \cdot w$ Now, take a figure F on a plane in three dimensions of area A; dragging it out in a direction perpendicular to the plane for a distance h, we get the *prism* with base F . Following the preceding logic, the measure of this solid figure (called its *volume*) is the product of h with A: $V = A \cdot h$. In particular, if the figure in the plane is the rectangle of side lengths l and w, the then solid figure (called a *rectangular prism*) is of volume $V = l \cdot w \cdot h$. If the figure we started with was a triangle of base b and altitude a, then the solid figure (called a *triangular prism* or *wedge*) has volume $V = \frac{1}{2}abh$.

In 7th grade we solved problems for prisms and cones based on polygonal figures. Now, suppose we start with a circle in the plane of radius r, and therefore of area πr^2. Drawing it out, the solid figure we get is a *circular cylinder* and its volume is $V = \pi r^2 h$: that is, the product of the area of the base with the height. It is customary to drop the adjective "circular", and call this the *cylinder* (see Figure 4).

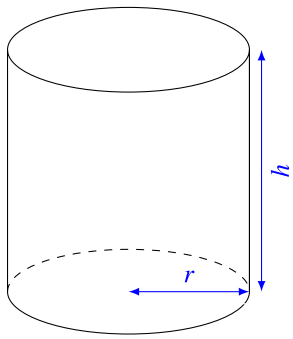


Figure 4

Example 15. Volume computations

First, as a reminder, let's begin with a review of seventh grade problems.

a. Find the volume of a prism built on a rectangle of side lengths 50 feet and 18 feet, and of height 25 feet.

Solution. The volume is the product of these lengths, so is $50 \times 18 \times 25 = 22500$ cubic feet. If you remember that 50 is half a hundred and 25 is one-fourth a hundred, this makes the computation easier:

$$50 \times 18 \times 25 = \frac{1}{2}10^2 \times 18 \times \frac{1}{4}10^2 = \frac{1}{8} \times 18 \times 10^4 = 2.25 \times 10^4 = 22500 \, .$$

b. A *wedge* is a triangular prism whose base is a right triangle of side lengths 2 in by 5 in, and whose height is 8 in. What is the volume of the wedge (as shown in Figure 5)?

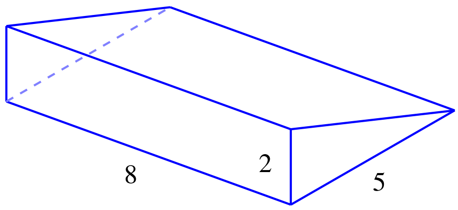


Figure 5

Solution. We see this as a 2×5 right triangle dragged out 8 inches, so the area is

$$(\frac{1}{2}2 \cdot 5) \cdot 8 = 40 \text{ cubic inches} \, .$$

c. A construction company wants to build a small shed covering a rectangular plot, that is 10 feet high, 18 feet long and has 3600 cubic feet f volume, What should its width be?

Solution. The shed is a rectangular prism of 3600 cubic feet, its height is 10 feet and its length is 18 feet. If we let w represent the width, we must have $18 \times w \times 10 = 3600$, or $180w = 3600$. That does it! So $w = 3600/180 = 20$ feet.

d. An ice cream company wants to package a pint of ice cream in a circular cylinder that is 4 inches high. What does the radius of the base circle have to be?

Solution. A pint is 16 fluid ounces,but we need this in cubic inches. We search and find that 1 fl oz is 1.8 cu. in. Now we can proceed. A pint is 16 fl oz, so is $16 \times 1.8 = 28.8$ cu. in. We want to put this in a cylinder of height 4 in, and of base radius r, and we want to find the value of r. What we know is that $\pi r^2 h = V$, and we know $V - 28.8$, $h = 4$ and let's take $\pi = 3.14$. We have to solve the equation

$$(3.14)r^2(4) = 28.8 \quad \text{or} \quad r^2 = \frac{28.8}{4(3.14)} = 2.29\ .$$

Since $1.5^2 = 2.25$, we can conclude that the radius is slightly more the one and a half inches.

In seventh grade we introduced Cavalieri's principle in order to explain facts about volume calculations, on the grounds that this principle is intuitively plausible, although not technically part of the middle school curriculum. We shall once again base our understanding on this mathematical tool.

Cavalieri's principle: Suppose that we stand two figures side by side. Suppose that every horizontal slice through the two figures gives two planar figures of the same area. Then the volume of the two solid figures is the same.

Notice that we do not require that the figures in the sections have the same size and shape, only that they have the same planar area. This more general interpretation will be useful to us in studying curved solids in three dimensions. We also want to notice that Cavalieri's principle applies in two dimensions (where the section is made by a line parallel to the base, and the hypothesis is that the lengths of the line segments are the same. This gives us another explanation of the formula $V = \frac{1}{2}bh$ for the area of a triangle, where b is the length of the base of the triangle, and h is its height (see Figure 6).

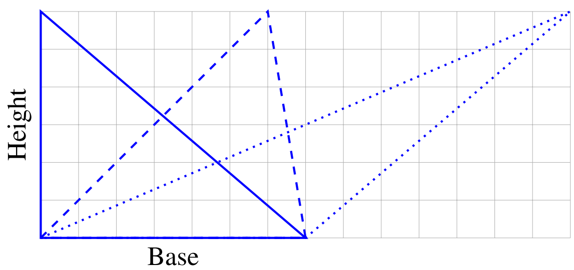


Figure 6

Figure 7 illustrates the reasoning behind Cavalieri's principle, where the individual figures could represent rectangles in the plane, or rectangular prisms or cylinders in three dimensions.

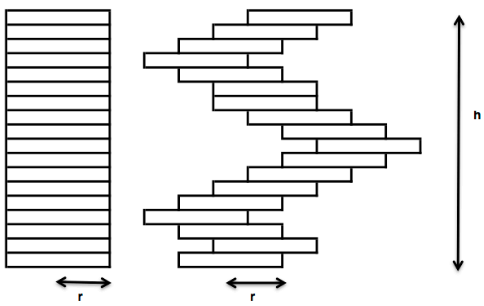


Figure 7

Example 16.

A buttress to a wall is a support structure that extends out from the wall to the ground, as in Figure 8. That buttress is made of blocks, all with base area of 3 sq. ft. The top of the buttress lies against the wall 30 feet above the ground. What is the volume of the buttress?

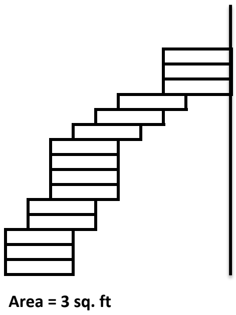


Figure 8

SOLUTION. Using Cavalieri's principle, we can consider this as a column of bricks of area 3 sq. ft. that rises 30 ft. above the ground. So, the volume is $Bh = 3 \times 30 = 90$ cu. ft.

Cones

Recall from 7th grade that a *cone* is a three dimensional shape consisting of a figure in the plane (called the *base* B), a point A not on that plane (called the *apex*) and all line segments joining A to a point on B.In 7th grade we discussed cones whose base is a polygon in the plane, here we turn to the *right circular cone*: a cone whose base is a circle in the plane (see the image on the right in Figure 9). Last year we discussed the formula for the volume of the pyramid on the left: the base is a square of side length a and the height is h: $V = \frac{1}{3}a^2h$: the volume of the pyramid is one-third the product of the area of the base and the height. This formula is true for every cone, in particular the right circular cone:

Volume of a Cone: The volume of a right circular cone is one-third the product of the area of the base and the height. If the height is h and the radius of the base is r, then $V = \frac{1}{3}\pi r^2 h$.

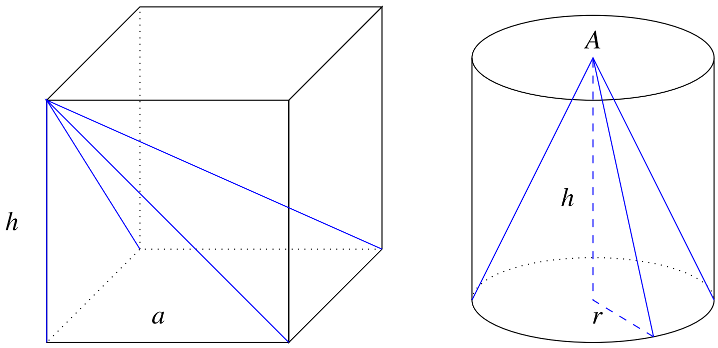


Figure 9

We can verify this fact by the following experiment. Select a cone and a cylinder of the same base radius r and height h; make sure that the cone is closed at its vertex and open at its base. Fill the cone with water and pour that water into the cylinder.The volume of the cone is the volume of the column of water in the cylinder. Depending upon how carefully the measurements are made, it will turn out that the height of that cylinder is $\frac{1}{3}h$, and so the volume of the cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$.

In 7th grade we saw how to fill the cube on the left with three copies of the pyramid shown on the left in Figure 9, thus confirming this formula. The ancient Greeks took this to show that the 1/3 was the right factor for all cones, and tried in vain to find a way to fill a cylinder with three copies of the circular cone it circumscribes. It was not until the Calculus was developed that this ”1/3” was understood. Actually, about a century earlier, Cavalieri showed how to effectively approximate the volume of a cone by slicing it into thin discs parallel to the base, noting that the radius of the discs decreases linearly, and then adding the volumes of the discs.

Example 17.

Let’s illustrate Cavalieri’s computation, by finding the volume of a tower of Hanoi (in the background on the left in Figure 10).

Figure 10

The tower of Hanoi is the stack of cylindrical discs in the corner behind the work desk. The radii of the discs increase linearly as we go down the tower. The game using the tower of Hanoi employs to the two vertical rods as well: the point being to move all the discs onto one of the other rods, one disc at a time so that no disc is ever placed on top of a smaller disc. The NCTM site, *Illuminations* has a java script to play the game (http://illuminations.nctm.org/Activity.aspx?id=4195). Our interest however, is just to try to calculate the volume of the entire tower. Now the game can be played with any number of discs; the tower in the photo has 32 discs. As the computation is tedious, we’ll work it out for 9 discs (the most common size for playing the game).

Let us take the radius of the base of the tower of Hanoi to be r units, and the height h units. Then, since there are 9 discs, the height of each disk is $h/9$. Since the radii of the of the disc increase linearly, and there are 9 of them, the radius of the top disk will be $1/9$, of the second disc, $2/9$ and so forth. To calculate the volume of the tower, we’ll work downwards from the top, adding one disc at a time. Now the first disc is a cylinder of base radius $r/9$ and height $h/9$, so its volume is

$$\text{Volume of top disc } = \pi(\frac{r}{9})^2\frac{h}{9} = \pi\frac{r^2h}{9^3} \ .$$

The radius of the second disc is $2r/9$, and its height is $h/9$ and we have:

$$\text{Volume of second disc } = \pi(\frac{2r}{9})^2\frac{h}{9} = \pi\frac{4r^2h}{9^3} \ .$$

The radius of the third disc is $3r/9$, and so The radius of the second disc is $2r/9$, and its height is $h/9$ and we have:

$$\text{Volume of third disc } = \pi(\frac{3r}{9})^2\frac{h}{9} = \pi\frac{9r^2h}{9^3} \ .$$

The pattern is clear: each time we move down a disc, the coefficient of r is the next integer over 9, and as the height is always $h/9$, we see that:

$$\text{Volume of the } k\text{th disc } = \pi(\frac{kr}{9})^2\frac{h}{9} = \pi\frac{k^2r^2h}{9^3} \ .$$

The volume of the tower of Hanoi is the sum of the volumes of the individual discs, and the volume of each disc is a factor of $\pi r^2 h$; the factor for the kth disc is $k^2/9^3$. Now, we do the calculation:

$$\frac{1^2 + 2^2 + 3^2 + \cdots 9^2}{9^3} = \frac{285}{729} = 0.391 ,$$

which is beginning to look a little like 1/3. What we will see, if we do what Cavalieri did, is that the more discs there are in the tower of Hanoi, the closer that factor gets to 1/3. In fact, for the 32 disc tower of Hanoi in the photo above, the factor of $\pi r^2 h$ is

$$\frac{1^2 + 2^2 + 3^2 + \cdots 32^2}{32^3} = \frac{11440}{32^3} = 0.349 ,$$

Were we to do this for a tower with 1000 discs, we'd find that the factor is 0.3338335 - getting really close to 1/3. Cavalieri did not just keep finding the sum; without computers even the first sum (up to 9) that we did would have been too difficult. He was smarter, he studied the algebraic form of the answer to be able to conclude that it gets closer and closer to 1/3 as the discs get thinner and the number of discs gets larger.

The Sphere

A *sphere* of radius r is the set of all points in space that are of a distance r from a point C, called the *center* of the sphere (see Figure 11).

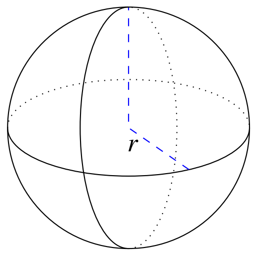


Figure 11

We can find a formula for the volume of a sphere with physical models as we did above for the cone. Pick a hemisphere (half of a sphere) of radius r, and a cylinder of radius r and height r. Now, fill the hemisphere with water and pour the water into the cylinder. Again, depending upon the care of measuring, we will find that the water level comes to about 2/3 the way to the top. So the volume of the column of water, and therefore, that of the hemisphere is $V = \pi r^2(\frac{2}{3}r) = \frac{2}{3}\pi r^3$. Since the hemisphere is half a sphere, we get

> The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

Example 18.

It fascinated the Greeks that, in terms of volume, a cylinder consisted of a cone and a hemisphere. They sought, without success, a constructive method to show that a cylinder can be decomposed into a cone and a hemisphere, or at least one that does not depend upon physical measurements. It was not until Calculus was invented that one found a mathematical proof of this fact; but we can use Cavalieri's principle to see why this is true. Place a cylinder, hemisphere and cone on a table as shown in Figure 12.

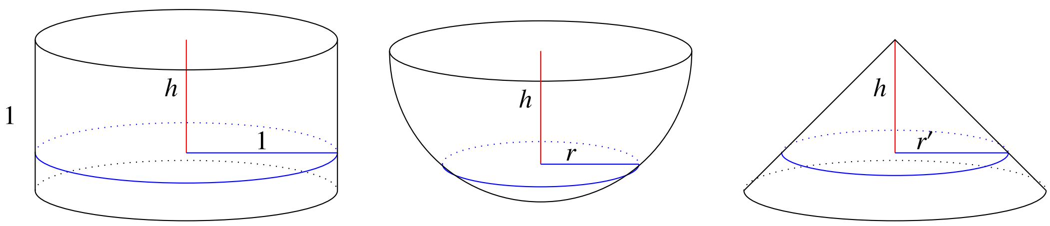


Figure 12

The height and base radius of the cylinder and the cone are both 1 unit, and the radius of the hemisphere is also 1. Now let us take a section of this setup by a plane parallel to the base and a distance h from the top of the figures.

The plane section of each figure is a circle, for the cylinder it is a circle of radius 1; let r be the radius of the circular section of the hemisphere, and r' that of the cone. If we can show that $r^2 + r'^2 = 1$, then by Cavalieri's principle we are done: the area of the section of the cylinder is equal to the sum of the area of the section of the hemisphere and the area of the section of the cone. Let's start with the hemisphere: by the Pythagorean theorem, $h^2 + r^2 = 1$, since the hypotenuse of that triangle is a radius of the sphere, of length one unit. Now for the cone: the triangle of sides labeled h and r' is isosceles, so $r' = h$. Thus $r^2 + r'^2 = 1 - h^2 + h^2 = 1$.

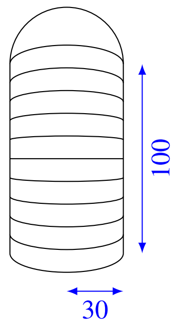
Example 19.

A farmer wants to raise 250,000 sq ft of wheat, and have it watered with a rotary irrigator. Approximately what should the radius of the circle be?

Solution. The area of the circle is πr^2, where r is the radius. So, we must solve $\pi r^2 = 25 \times 10^4$. Divide both side by π and use 3 as an approximation of π to get $r^2 = 8 \times 10^4$. Now $\sqrt{8 \times 10^4} = 2\sqrt{2} \times 10^2$, so, using 1.5 as an approximate value for $\sqrt{2}$, we get that the radius is about 350 ft.

Example 20.

The same farmer has a silo with a base radius of 30 feet and a storage height of 100 feet. A silo is a storage bin that is a cylinder with a hemisphere on top. The "storage height" is the part which can be filled with grain - it is just the cylinder. A cu. ft. of grain weights 62 lbs. How many pounds of grain can the farmer store in the silo?

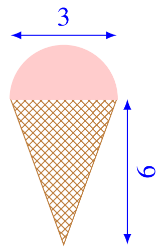


a. How high (including the hemispherical top) is the silo?

b. 1000 sq ft of wheat produces 250 lbs of grain. Is the silo large enough to hold the grain? By how much?

Example 21.

An ice cream cone consists of a cone filled with ice cream topped with a hemisphere of ice cream. If the cone is 4 inches long and the top has a diameter of 3 inches, how much ice cream (in cu in) fits in the cone. If 6 cu in of ice cream is equal to 1 fl oz, how many ounces of ice cream is that? If one fl oz of ice cream is 143 calories, how many calories is that?



Solution. We assume that the ice cream cone is completely full and is topped with a hemisphere of ice cream. Now, let us use the given data: the cone has height 4 in, and radius 1.5 in. The hemisphere has radius 1.5 in. Then the total volume is

$$\frac{1}{3}\pi(1.5)^2(4) + \frac{2}{3}\pi(1.5)^3 = \pi(\frac{1}{3}(2.25)(4) + \frac{2}{3}(3.375)) = 5.25\pi$$

cu. in. Using 3.14 as an approximation for π, this gives us 16.48 cu. in., or 2.75 fl. oz, since there are 6 cu. in. in a fl. oz. At 143 calories a fl. oz., the cone contains about 393 calories.

Chapter 9
Geometry: Transformations, Congruence and Similarity

By the third century BCE, the Greeks had gathered together an enormous amount of geometric knowledge, based on observations from the ancient Greeks (such as Pythagoras), ancient civilizations (Babylonian, Egyptian) and their own work. Aristotle and his successors set about the task to put this knowledge on a firm logical basis. A result of their work is the "Elements of Geometry" by Euclid (the name may be of one person or of the group). Here the foundation of the subject lay in a set of "self-evident" axioms, and "constructions" by straightedge and compass. For example, two figures were called *congruent* if it we could copy one onto the other with straightedge and compass. These tools were used to copy points, line segments, circles and angles. In particular, they did not use (numerical) measure:, for example, the measure of a line segment was the distance between the pins of the compass when put at the endpoints of the segment. From there, the point of Euclid's Elements is to deduce all current knowledge from these basics, using only these axioms and tools and Aristotelian logic. It was important that the logical structure not lay in folklore and structure, but on the axioms (although it is the folklore and constructions that convince us that these axioms are self-evident). Concepts such as "same shape" and "same size" were given explicit definition, all based on a small set of "undefined objects" (point, line, etc.) whose understanding was intuitive. An objective, of course, was to minimize the number of concepts that were to be understood intuitively; while everything else is understood by definition or strict logic. The truth of assertions is justified solely by the application of logic to already known truths, and not by construction, observation, and above all, not by techniques involving movement from one place to another.

This accomplishment was monumental, and formed the basis of geometric instruction for over 2000 years. We should point out that, almost immediately, philosophers began to object that (at least) one of the "self-evident" axioms was in fact, not so self-evident, and that was the axiom that dealt with parallelism. To paraphrase the problem, two lines are said to be parallel if they never meet. Now, all other axioms could be intuitively understood by pictures and constructions on a given piece of paper (or papyrus or the sand) of finite dimensions. This axiom, however, requires us to conceive of going however far we have to to show that two lines meet, and thus are not parallel, and worse: we can never verify that two lines *never meet*.

In the 19th century this dilemma was put to rest: there are planar geometries where all lines eventually intersect, and others for which almost all lines never intersect. These were respectively, spherical and hyperbolic geometry, and they were discovered because the applications of mathematics needed understanding of these geometries. In the late 19th century, the mathematician, Felix Klein formalized a new concept of geometry, broad enough to encompass all these forms. This geometry is based on its dynamical, rather than static, use. In Kleinian geometry, the primary concept is that of *transformation*: a set of transformations are specified, and geometry becomes the study of objects that do not change under these transformations. This is the approach that is adopted in mathematical instruction of today and is called *transformational geometry*. So for example, translations are allowable transformations and a triangle changes its position under a translation, but not its shape and size.

During the same time, other mathematical ideas were developing and maturing that would mesh with this geometric thread. The introduction of coordinates in the 17th century, and the development of linear algebra in the

19th century presented a rich set of tools in which to develop geometry; we can call this *coordinate geometry*. This subject provides a new perspective: for example, distance between two points is defined in terms of the coordinates of the points, and not in terms of a scale along a straight edge placed on the two points. This approach leads to a complete interpretation of geometry in terms of algebra and through this interpretation, a new way to rediscover geometry. The word *rediscover* is used deliberately, for coordinate geometry provides us with a way to precisely calculate measures in geometry, but not a new way to develop the subject. For example, in coordinate geometry, the Pythagorean theorem becomes the definition of distance. So, why is the Pythagorean theorem true? We still have to return to either Euclidean or transformational geometry and the fundamental constructions.

All of this will be developed in a systematic way in secondary mathematics. The objective in 8th grade is to give the students the opportunity of free exploration of the basic concepts of transformational geometry: rigid motions, dilations, congruence and similarity. In chapters 9 and 10 students will verify that rigid motions preserve the measures of line segments and angles, and that dilations preserve the measure of angles, while changing measures of line segments by a constant factor. From there we go on to observe basic geometric facts, some of which will be made explicit in the classroom, while others are discovered through the student's own work.

Our philosophy is that the understanding of much of secondary mathematics is dependent upon the strength of the students' geometric intuition, and that intuition is best developed through free play with the fundamental concepts and ideas. So, to some, our exposition may seem unstructured (for this reason we provide a structured appendix); we hope the teacher will see it as overstructured, and whenever possible, to proceed in the direction the class takes, even if it veers too much from the text.

In Chapter 2, section 2, students learned that the slope of a line can be calculated as rise/run starting with any two points on the line. To show why this works, we introduced translations and and dilations, and observed their properties as transformations of the plane. We used these basic properties of dilations: it has one fixed point (the center of the dilation) and all other points are moved away (or toward) the center. There is a positive number r such that the dilation multiplies any length by r. Note that if $r = 1$, then there is no movement at all: in this case the dilation is called the *identity*: no point moves.

In this chapter we begin to look at transformations of the plane more deeply, in order to get an understanding of the shape and size of a geometric object, no matter where it is positioned on the plane. Students have already seen *shifts, flips* and *rotations*: here we reintroduce them as motions of the plane that preserve the basic geometric measures: that of angles and lengths. In discussing these motions, and dilations, one should take a dynamic, not static approach. We are not picking up an object and dropping it, we are "moving" it to its new location.

A *rigid motion* of the plane is a transformation of the plane that takes lines to lines, and preserves lengths of line segments and measures of angles. That is, under a rigid motion, a line segment and its image have the same length, and an angle and its image have the same measure. An example of a rigid motion is a *translation* (called a *shift* until now). There are two other basic kinds: *reflections* (flips) and *rotations* (turns). Now we consider two figures *congruent* (of the same shape and size) if there is a sequence of rigid motions that takes one to the other. This is a different way of looking at the equivalence of two figures without changing the meaning: if two objects are congruent by way of a Euclidean construction, then there is a sequence of rigid motions that takes one to the other. And if we can move one object onto another by rigid motions, there is a construction taking one to the other. The advantage of working with motions rather than constructions, is that the idea is more directly related the the use of geometry in science and engineering: one does not put a beam on a house by construction in place, but by moving the beam from one place to the other. If we want to create a robot to do that job, we need to conceive of it in terms of rigid motions, not constructions.

In the second section we turn to dilations and scale factors: a dilation preserves lines and angles, but changes the scale of length of line segments. We say that two figures are *similar* (have the same shape) if there is a combination of rigid motions and dilations that takes one to the other.

The focus of 8th grade geometry is to explore the concepts of transformations, congruence and similarity by experimenting with them and gaining familiarity with the correspondence between constructing a new image of an object, and moving the object to its new location. We concentrate on the "what" and "how" of geometry, while high school geometry extends that basis to understanding the "why." In real-life science and industry, people almost constantly draw representations (called *graphics*) of their work, even if it is about medical procedures or finance rather than architecture or construction. In 8th grade we plant the foundations for these skills.

To begin with, students should be given an opportunity to discuss the concepts of “same shape” and “same size and shape.” This is the purpose of the following example, and many of the preliminary exercises in the workbook.

Figure 1 shows several sets of objects. In Figure A all the images are of the same size and shape, and we can move any one to any other one by a rigid motion. In the remaining figures, there is no rigid motion taking one figure to the other. Try to understand how to move the first object in Figure A on the others. Why can’t this be done for the other sets of objects? Note that in Figure B, the figures are of the same shape, but not of the same size, and in Figures C and D the figures are neither of the same size nor shape.

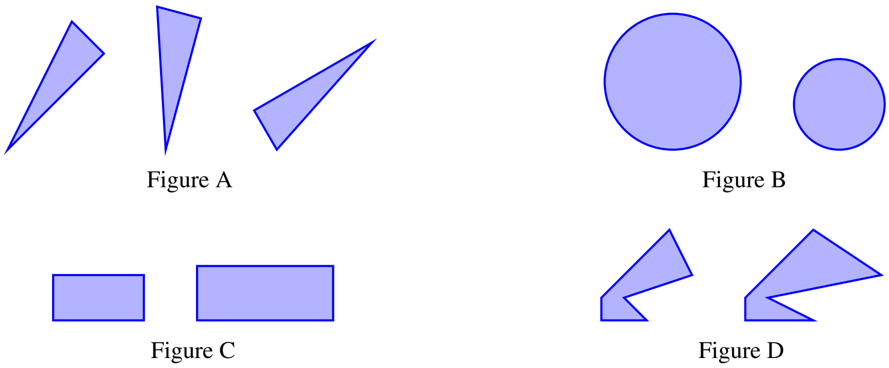


Figure 1

Let’s recall some basic geometric facts that have been observed in previous grades.

1. A line is determined by any two different points on the line, by placing a straight edge against the points and drawing the line.
2. Two lines coincide (are the same line) or intersect in precisely one point or do not intersect at all. The issue may come up: what if they do not intersect on my paper, how do I know whether or not they ever intersect? Because the question did come up in the days of Euclid, it generated a controversy that lasted for almost 2000 years.
3. Two circles do not intersect, or intersect in a point, or intersect in two points. If they intersect in more than two points, they actually coincide.
4. Two lines that do not intersect are said to be *parallel*. If two lines intersect and all the angles at the point of intersection have the same measure, the lines are said to be *perpendicular*.
5. The sum of the lengths of any two sides of a triangle is greater than the third.

Section 9.1. Rigid motions and Congruence

Understand congruence in terms of translations, rotations and reflections, (rigid motions) using ruler and compass, physical models, transparencies, geometric software.

Verify experimentally the properties of rotations, reflections and translations:

a) lines are taken to lines, and line segments to line segments of the same length;

b) angles are taken to angles of the same measure;

c) parallel lines are taken to parallel lines. 8G1

A rule that assigns, to each point in the plane another point in the plane is called a *correspondence*. Often a correspondence is defined in terms of coordinates, and written this way: $(x, y) \rightarrow (x', y')$, where the values of x', y' are given by the rule, which may be a formula or a set of instructions. In the case of a formula, we call it the *coordinate rule*

Example 1.

Where possible give the coordinate rule for the correspondence.

- **a.** Move a point P to a point P' on the same ray through the origin that is twice the distance from the origin.
- **b.** Move every point in the plane 3 units to the right and 1 unit down.
- **c.** Multiply the first coordinate by 2 and the second coordinate by 3.
- **d.** Replace each coordinate of the point by its square.
- **e.** Interchange the two coordinates. How do we describe the transformation in geometric terms?

Solution.

- **a.** If (x, y) are the coordinates of the point P, then the coordinates of any point on the ray from the origin through P is of the form (rx, ry) for some $r > 0$ (for a line through the origin represents a proportional relationship). In our case $r = 2$, so the coordinate rule is $(x, y) \rightarrow (2x, 2y)$.

- **b.** If (x, y) are the coordinates of a point P, then the coordinates of the point three units to the right and one unit down are $(x + 3, y - 1)$. So the coordinate rule is $(x, y) \rightarrow (x + 3, y - 1)$.

- **c.** Here the coordinate rule is $(x, y) \rightarrow (2x, 3y)$.

- **d.** The coordinate rule is $(x, y) \rightarrow (x^2, y^2)$.

- **e.** The coordinate rule is $(x, y) \rightarrow (y, x)$. This can be described as a flip (reflection) in the line $y = x$.

A *mapping* (or *transformation* T of the plane is a correspondence that has the property that different points go to different points; that is, for two points $P \neq Q$ we must also have $T(P) \neq T(Q)$. This is in fact just what a map does: it takes a piece of the surface of the earth and represents it, point for point, on the map M. When we study the effect of a mapping on objects, it is useful to call the object K the *pre-image* and the set of points to which the points of K are mapped is the *image*, denoted $T(K)$. Of the rules described in example 2, rules 1,2,3,6 are mappings, while rules 4 and 5 are not. Before going on, a little more vocabulary. An object is *fixed* under a mapping, if the mapping takes the object onto itself. When we say that an attribute is *preserved* we mean that if the original object has that attribute, so does the image object.

To be useful, geometrically, a transformation must preserve features of interest: so for example, a scale drawing of an object changes only the dimensions, and not the shape, of the object. In this section we are interested in mappings that preserve both the dimension and shape of objects. These are the *rigid motions*. These are mappings of the plane onto itself that takes lines to lines and preserve lengths of line segments and measures of angles. That definition is the starting point of geometry in Secondary 1, but not in 8th grade, where the emphasis is on an intuitive understanding of rigid motions and their action on figures. So, for us, rigid motions are introduced by visualization and activities. Take two pieces of transparent paper with a coordinate grid. Place one on top of the other so that the coordinatizations coincide. A *rigid motion* is given by a motion of the top plane that does not wrinkle or stretch the piece of paper.

Given a figure on the plane, we can track its motion by a rigid motion T in this way. Place a piece of transparent graph paper on top of another so that the coordinate axes coincide. We start with a figure on the bottom piece of paper. Copy that figure on the upper piece of paper. Enact the rigid motion T by moving the upper piece of paper. Now trace the image on the upper piece of paper onto the lower. This figure is the result of moving the given figure by the rigid motion T. Keep in mind that the lower piece of paper is where the action is taking place; the upper piece is the action. Students should experiment with these motions using transparent papers. In the process they will observe that these motions do preserve lines and the measures of line segments and angles. If a figure doesn't change; that is T takes the figure onto itself, we say that the figure is *fixed*. Students may observe that there are three fundamental kinds of rigid motions characterized by: a) no point is left fixed b) precisely one point is left fixed, c) there is a line all of whose points are left fixed.

Rigid motions include

- **Translation**: these are the rigid motions T of the plane that preserve "horizontal" and "vertical": that is, horizontal lines remain horizontal, and vertical lines remain vertical. We can describe this as sliding the top piece of paper over the bottom so that the horizontal and vertical directions remain the same.
- **Reflection** in a line. Select a line on the plane, this will be the *line of reflection*. The line separates the plane into two pieces. For a point in one of the pieces, the reflection moves it to the point on the other piece whose distance from the line is the same as the distance of the original point on the line. All points on the line remain in the same place. We effect this with the transparent paper in this way: fold the top transparency (in three dimensions) along that line so that the sides determined by the line exchange places, and so that nothing on the line moves.
- **Rotation** about a point: these are the rigid motions of a plane that leave one point fixed. This point is called the *center of the rotation*. The center will not move. All other points are moved along a circle with that point as center.

Through experimentation with these movements, students should observe that translations do not leave any points fixed; rotations are rigid motions that leave just one point fixed; for a reflection, all the points on the line of reflection remain fixed. In the next subsections we will work with these motions in detail and collect together their properties.

Example 2.

Of the rules in Example 1, which can be represented by a rigid motion?

Solution.

a. This is not a rigid motion: objects change size. Furthermore, it is a transformation that has one fixed point, but it is not a rotation. In particular. it cannot be illustrated using transparent papers, for it stretches the top paper.

b. This can be realized as a rigid motion: shift the top piece of paper so that the the origin goes to the point $(3, -1)$.This is an example of a translation.

c. This is like a) but even more complicated: the horizontal stretch is 2, and the vertical stretch is 3.

d. The squaring rule does not take distinct points to distinct points: for example (1,1) and (-1,-1) both go to (1,1). In fact, for any positive a and b, all four points $(a, b), (-a, b), (a, -b), (-a, -b)$ go to the same point (a^2, b^2).

e. This transformation is realized by reflection in the line $y = x$.

Translations

We have defined *translation* as a rigid motion of the plane that moves horizontal lines to horizontal lines and vertical lines to vertical lines. Let's find the coordinate rule for a translation. Starting with a particular translation, let (a, b) be the coordinates of the image of the origin. We want to show that the pair of numbers (a, b) completely determines the translation; in fact, the coordinate rule is: $(x, y) \to (x + a, y + b)$. For a dynamic visualization, go to http://www.mathopenref.com/translate.html.

For any point (x, y), draw the rectangle with horizontal and vertical sides with one vertex at the origin and the other at (x, y) (see Figure 2). The translation takes this rectangle to a rectangle with horizontal and vertical sides with one vertex at (a, b) and the other at the image of (x, y). Since the lengths of the sides are preserved, the image rectangle has the same dimensions as the pre-image, and so the vertex across the diagonal from (a, b) has to be $(x + a, y + b)$. But that is the image of (x, y).

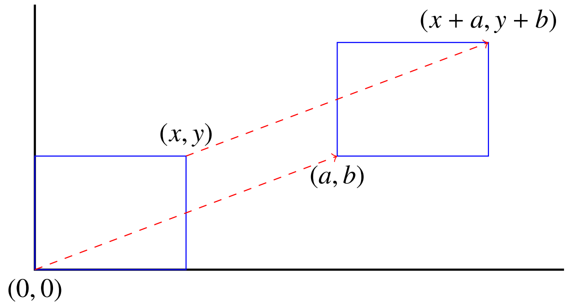


Figure 2

We refer to (a, b) as the *vector* for the translation, for it shows both the direction in which any point is translated, and also the distance it s translated. Now, in our figure, the point (x, y) was chosen to be in the first quadrant; but the same reasoning works for any point (x, y).

This reasoning also shows that a translation preserves the slopes of lines (in particular, any line and its image are parallel). In Figure 3 we show the effect of a translation on the line L, denoting its image by L'. Draw the slope triangle AVB for the line L as shown. Now, the translation moves that triangle to the triangle $A'V'B'$ as shown. Since the translation preserves "horizontal" and "vertical," $A'V'B'$ is a slope triangle for the line L'. Now since the translation preserves lengths, the horizontal and vertical legs of the slope triangle on L' have the same lengths as the horizontal and vertical legs of the slope triangle on L, so the lines have the same slope. Figure 3 is deliberately drawn so that L and L' are not parallel in order to show what goes wrong.

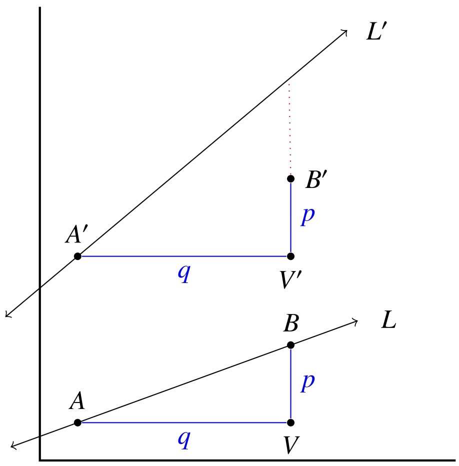


Figure 3

Properties of translations:

- A translation preserves the lengths of line segments and the measures of angles.
- For a translation, there is a pair (a, b), called the *vector* of the translation, such that the image of any point (x, y) is the point $(x + a, y + b)$.
- Under a translation, the image of a line L is a line L' parallel to L. Furthermore, translations take parallel lines to parallel lines. This is because a translation does not change the slope of a line.
- A translation that does not leave every point fixed does not leave any point fixed.

Reflections

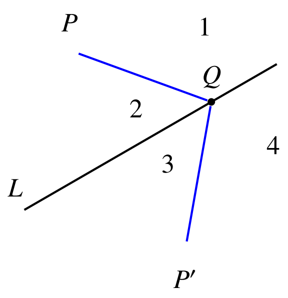


Figure 4

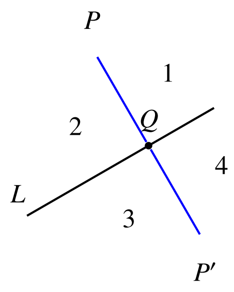


Figure 5

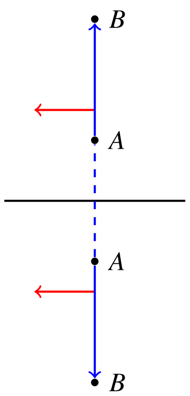


Figure 6

We have defined a *reflection* by the action of flipping (or folding) along a line L, called the *line of reflection*. A reflection can be described this way: it is a motion that leaves every point on L fixed, and for a point P not on L, with P' its image under the reflection, L is the perpendicular bisector of the line segment PP'. We now show that reflections as we visualize them (folding the plane along the line L) have these properties. For a dynamic visualization, go to http://www.mathopenref.com/reflect.html.

First, it is clear, since it is described by a motion that does not stretch our paper in any direction, that a reflection preserves the lengths of line segments and the measure of angles. Let L be the line of reflection for the reflection T, let P be a point on one side of L and P' the image of P under the reflection. Draw any line from P to a point Q on L. Let P be the image of the point P under the reflection. This situation is depicted in Figure 4.

Since a reflection takes lines to lines and leaves the point Q fixed, the image of the segment PQ is the segment $P'Q$. Since the lengths of these segments are the same, and Q was chosen as any point on L, we conclude that the points P and P' are at the same distance from any point on the line L. Furthermore the image of $\angle 1$ is $\angle 4$, so they have the same measure, and the image of $\angle 2$ is $\angle 3$, so they have the same measure. In particular, if Q is chosen so that PQ is perpendicular to L (and thus all angles at Q are the same), since we already know that the segments PQ and $P'Q$ have the same length, we conclude that L is the perpendicular bisector of PP'.

Something special happens with reflections that does not happen with other motions. Notice that for translations and, as we will see, rotations, we do not have to lift the top piece of transparent paper off the bottom piece; but with reflections we must do so; we execute what we can call a *flip* over the line of the reflection. This has an important effect that is not shared with translations and rotations. Suppose that A and B are two different points on a line perpendicular to the line of reflection L. Consider that line as directed from A to B, and at A draw a little line segment on the left side of the directed line segment AB. Now reflect this configuration in the line L. The little line segment now lies on the right of the directed image line segment $'AB'$. Another way of putting this is that the rotation from the direction of AB to the direction of the small red arrow is counterclockwise in the original, but clockwise in the image. Since the point in question could have been any point in the plane, what has happened is that the reflection changed the sense of clockwise to counterclockwise everywhere This sense of rotation, clockwise or counterclockwise, about any point is called *orientation*.In short, a reflection changes the orientation of the plane.

Orientation can also be described in terms of angles. Consider an angle $\angle AVB$ (with vertex V) determined by the rays VA and VB. Looking out at the angle from V we can say that one of the rays is *clockwise* from the other (in Figure 7A, the ray VB is clockwise from VA). The reverse direction is called *counterclockwise*. Now the point we want to make is that reflections interchange the rays of an angle in the sense of orientation. This is depicted in Figure 7B, representing a reflection in the line L. The angle $\angle AVB$ goes to the angle $A'V'B'$ under the reflection. But, while the ray VB is clockwise to the ray VA, the image ray $V'B'$ is counterclockwise to the image ray $V'A'$. This is what happens when we look in a mirror: our left hand is on the right side of that person in the mirror.

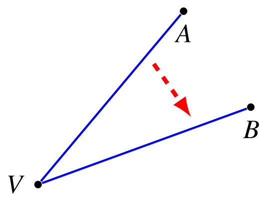


Figure 7A

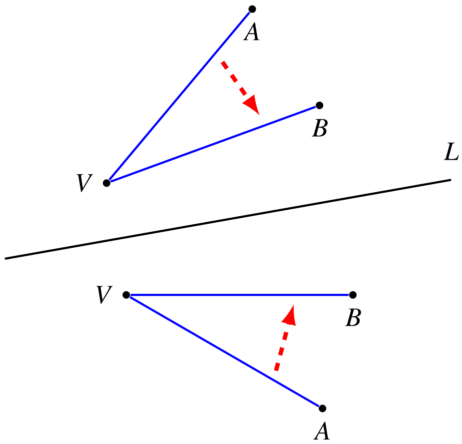


Figure 7B

Properties of Reflections:

- A reflection preserves the lengths of line segments and the measures of angles.
- For a reflection, there is a line L, called the *line of the reflection*, such that for any point P, L is the perpendicular bisector of the line segment joining P to its image.
- A reflection leaves every point on L fixed, and interchanges the two sides of that line. If the image of a point P is P', then, for any point Q on L, the line segments PQ and $P'Q$ are the same.
- A reflection reverses orientation; that is if two rays start at the same point, and ray 2 is clockwise from ray 1, the the image of ray 2 is counterclockwise from that of ray 1.

It is possible to describe all rigid motions by coordinate rules; at this time it is most useful to just do this for particular special cases.do so for these particular reflections: in the coordinate axes and in the lines $y = x$ and $y = -x$.

Example 3.

Find the coordinate rule for the reflections **a.** the x-axis, **b.** the y-axis, **c.** the line $y = x$, **d.** the line $y = -x$.

Solution.

a. Reflection in the x-axis leaves the x coordinate the same and changes the sign of the y coordinate. For this reflection takes vertical lines to vertical lines, and so the x-coordinate is fixed. For any point (x, y) (with $y \neq 0$), its image is a point (x, y') with $|y| = |y'|$ and $y \neq y'$; the only possibility is that y' is $-y$.

b. Reflection in the y-axis leaves the y coordinate the same and changes the sign of the x coordinate. This has the same argument as for part **a**.

c. Reflection in the line $L : y = x$ exchanges the coordinates: a point (x, y) goes to the point (y, x). To show this, let's start with a point (a, b) not on the line, and thus $a \neq b$. See Figure 8 for the setup. Now draw the horizontal and vertical lines trom (a, b) to the line L. The horizontal line ends at (b, b) and the vertical line at (a, a). When this triangle gets reflected in the line L, the sides of the image line will consist of a vertical line segment from (b, b) and a horizontal line segment fro (a, a). The point of intersection of these lines has the coordinates (b, a) and is the image point of (a, b).

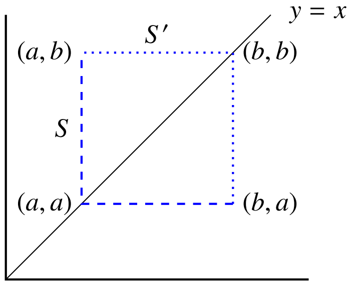


Figure 8

d. Using the same argument, but being super careful about signs, we can show that reflection in the line $L : y = -x$ can be described in coordinates as $(x, y) \to (-y, -x)$.

Rotations

We have defined a *rotation* as a rigid motion that turns a figure about a fixed point, called the *center* of the rotation. Since lines are mapped into lines and the center C is fixed, any ray with endpoint C is moved to another ray with endpoint C. A rotation can be defined by this property: the angle between any ray with endpoint C and the image of that ray always has the same measure α.

This can be shown using Figure 9: C is a point on the plane, and we are considering a rotation R with center C. Let A be a point on the horizontal ray from C to the right, and draw its image point A". As in the figure, denote the angle between the rays CA and CA' by double arcs. This is the angle of the rotation. Now take another typical point B, and denote its image point B'. Then the angle between the ray CB and CB' is also the angle of rotation, so it can be indicated by the double arch. Label the angles $\angle 1,\ \angle 2,\ \angle 3$ a shown in Figure 9. What we have seen is that

$$\angle 1 + \angle 2 = \angle 2 \angle 3,$$

so $\angle 1 = \angle 3$. But $\angle 1$ is the angle between the ray CA andCB and $\angle 3$ is the angle between the ray CA' and CB', so is the image angle , and thus we have shown that rotations preserve the measure of angles.

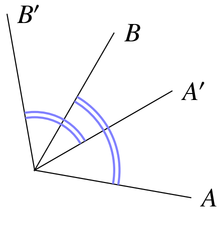


Figure 9

For a dynamic visualization of this discussion, go to http://www.mathopenref.com/rotate.html.

Properties of rotations:

- A rotation preserves the lengths of line segments and the measures of angles.
- For a rotation, there is a point C, called the *center of the rotation*, and an angle α called the *angle of rotation*. For any point P with image Q, the angle $\angle PCQ = \alpha$.
- A rotation preserves orientation; that is, if two rays start at the same point, and the second is clockwise from the first, then the image of the second is also clockwise from the first.

Example 4.

Find the coordinate rule for a rotation of **a.** 90^{circ} counterclockwise, **b.** 90^{circ} clockwise (usually denoted by -90^{circ}), **c.** 180^{circ}, **d.** -180^{circ}.

Solution.

a. See Figure 10, where P is a point in the first quadrant and Q is the image of P. The rotation moves triangle I into the position of triangle II. Note that the horizontal leg of triangle I corresponds to the vertical leg of triangle II, and the vertical leg of triangle I corresponds to the horizontal leg of the image, triangle II. The lengths of corresponding line segments are the same, but since the image is in the second quadrant the first coordinate is negative so we must have $(x, y) \to (-y, x)$ as labeled. This argument works as well no matter in which quadrant we start with the point P.

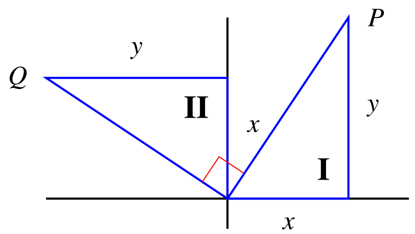


Figure 10

b. The same argument works, only now use Figure 11, and we conclude that the coordinate rule for a clockwise rotation by a right angle (of - 90°) has to be $(x, y \to (y, -x)$

Take a moment to note that a rotation by 90° takes a line into another line perpendicular to it. We saw in Chapter 2 that the product of the slope of a line and that of its image under a rotation by 90° is -1. Note that this is demonstrated in figure 10 and 11: the slope of the original line is y/x and that of its image is $-x/y$.

c. See figure 12, where P again is in the first quadrant and Q is its image. Since the rotation is by 180°, P and Q lie on the same line through the origin, and the length of the segment CP and CQ are the same. In other words, Q is diametrically opposite to P, so is the point $(-x, -y)$. Another way of seeing this is to recognize that a rotation by 180° is a rotation by 90° followed by another rotation by 90° . Now rotation boy 90° interchanges the coordinates and puts a minus sign in front of the first one. Thus the succession of two 90° rotations can be written in coordinates by

$$(x, y) \to (-y, x) \to (-x, -y) .$$

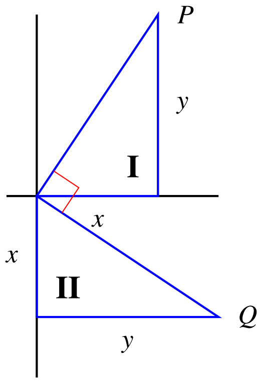


Figure 11

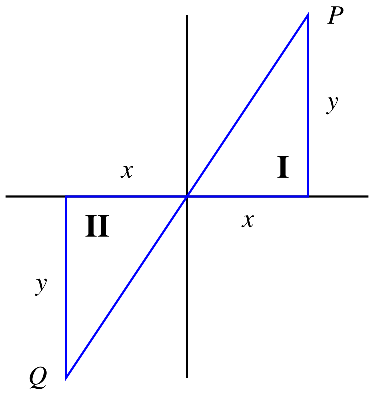


Figure 12

d. The same argument holds for a rotation by -180° , so is also given by the coordinate rule $(x, y) \rightarrow (-x, -y)$.

Succession of rigid motions

Now, a rigid motion is a transformation of the plane that takes lines into lines and that preserves lengths of line segments and measures of angles. If we follow one rigid motion by another,we get a third motion which clearly has the same properties: lines go to lines and measures of line segments and angles do not change. We have discussed specific kinds of rigid motions: translations, reflections and rotations. It is a fact that every rigid motion can be viewed as a a succession of motions of one or more of these types; in this section we will look at such examples.

Example 5.

Given two line segments AB and $A'B'$ of the same length, there is a rigid motion that takes one onto the other.

Solution. Figure 12A shows the two line segments, and indicates that we can translate the point B to the point B', getting the picture in Figure 12B. Now rotate the line segment AB' around the point B' through the angle $\angle AB'A'$, so that the segments AB' and $A'B'$ lie on the same ray. But since the segments have the same length, the point A lands on A', and the succession of the translation by the rotation is the rigid motion taking AB to $A'B'$.

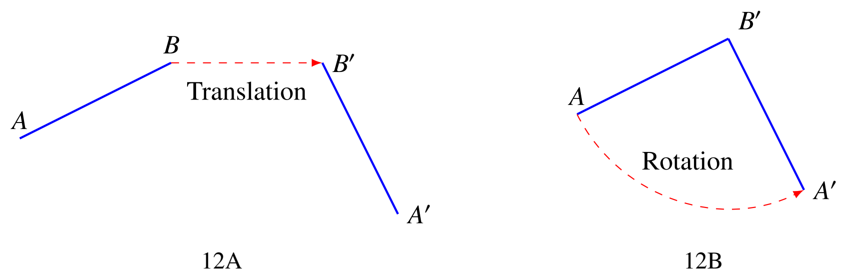


Figure 12

Example 6.

Given two circles of the same radius, there is a rigid motion, in fact, a translation, taking one circle onto the other. In fact, the translation of one center to the other does the trick. Can you explain why? Hint: use the definition of "circle."

Example 7.

If two angles have the same measure, there is a rigid motion of one to the other.

Solution. Let AVB and $A'VB'$ be the two angles. Notice, that the two angles have the same vertex. Initially, they need not, but a translation of one vertex to the other arranges that. Now, look at figure 13. We have already taken another liberty: we have labeled the rays of each angle so that the orientation is consistent: VB is clockwise from VA and VB' is clockwise from VA'. (If this wasn't the case originally, how can we make it so?) Now rotate with center V so that the ray VA lands on the ray VA'. Since the original angles had the same measure, and we have set up the orientation correctly, the ray VB falls on the ray VB'. The combination of the translation and rotation is the rigid motion landing one angle onto the other.

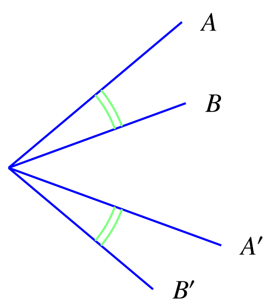


Figure 13

Example 8.

Under what conditions can we find a rigid motion of one rectangle onto another?

Solution. . First of all, rigid motions preserve lengths and angles, so any rigid motion will alway move a rectangle to another rectangle whose side lengths are the same. So, if there is a rigid motion of rectangle R onto rectangle R', the lengths of corresponding sides must be the same.

But now -to answer the question - if this condition holds, then there is a rigid motion of one rectangle onto the other. We will show this using Figure 14 of two rectangles $ABCD$ and $A'B'C'D'$ with corresponding sides of equal lengths. We have labeled the vertices so that the routes $A \to B \to C \to D$ and $A' \to B' \to C' \to D'$ are both clockwise. (Can you check that we can really do that?). By example

7, we can find a rigid motion taking the angle DAB onto the angle $D'A'B'$ (translate A to A' and then rotate). Since the lengths of corresponding sides are equal, that tells us that D lands on D' and B onto B'. Since both figures are rectangles, that forces C onto C', so the rectangles are congruent.

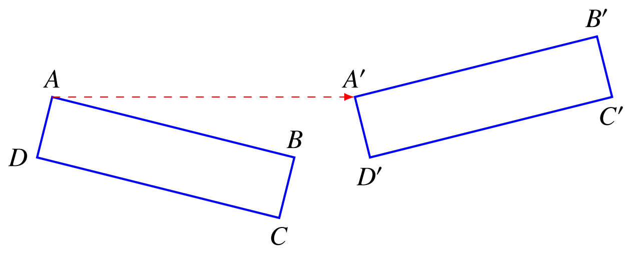


Figure 14

Example 9.

Since reflections reverse the orientation on the plane, the succession of two reflections preserves orientation, so has to be a pure rotation or a pure translation or a combination of the two. How doe we know when we get a translation, or when we get a rotation?

Solution. Let R be the reflection in the line L, S the reflection in the line L', and T the combined motion: R followed by S.

a. If the lines L and L' are the same line, then S just undoes what R did, and the succession of R by S leaves every point where it is. So, in this case, the succession of reflections is the *identity* motion - no motion at all.

b. Suppose that the lines L and L' are different, and intersect in a point C . Then C is fixed point for each, so is fixed under the succession T. Suppose T had another fixed point P. Then the first reflection (R) sends P to another point P' and S returns P' to P. Well, that means that P and P' are reflective images in both L and L', and that can only be if the lines are the same. Since they are not, T has only one fixed point: C. But the only motions with one fixed point are the rotations.

c. Now, if L and L' are different lines and have no point of intersection, then they are parallel. In this case, the transformation T formed by the succession of the two reflections has no fixed points. For if $T(P) = P$, this tells us that S takes $R(P)$ back to P, but the only reflection that does that is R. Since the lines are different, $S \neq R$, so there is no point P such that $T(P) = P$; that is there are no fixed points. Only translations are without fixed points, so T is a translation.

Example 10.

Let R be the reflection in the y axis, and S, reflection in the x axis.

a. Describe the rigid motion T defined by the succession: R followed by S?

b. Do the same problem with either of the lines of reflection replaced by the line $y = x$.

Solution.

a. Reflection in the y-axis changes the sign of the first coordinate, and reflection in the x-axis changes the sign of the second coordinate. So, the effect of both is to change the sign of both coordinates. The coordinate rule, then, for R followed by S (or the other way around) is $(x, y) \to (-x, -y)$

b. Now, let's call U reflection in the line $y = x$. That effect is to interchange coordinates. So, R followed by U is given, in coordinates by: $(x, y \to (-x, y) \to (y, -x)$, which is rotation by -90°. U followed by S is, in coordinates, given by: $(x, y) \to (y, x) \to (y, -x)$. The same answer!

Suppose we interchange the orders of R, S and U, do the answers change?

Example 11.

Let R be reflection in the line $x = 1$, and S reflection in the line $x = 2$. Describe the translation T, the succession R followed by S?

Solution. Both R and S take vertical lines to vertical lines, and take horizontal lines to themselves, so that is true of the succession T. Now, R takes (0,0) to (2,0) and S leaves (2,0) alone, so T takes (0,0) to (0,2). Since a translation does to all points what it does to one, we can say, in coordinates, that T takes (x, y) to $(x + 2, y)$.

Properties of a succession of two rigid motions:

- Rotations: if they have the same center then the succession of the two is a rotation with that center, and whose angle is the sum of the angles of the two given rotations.
- Translations: if T is a translation by (a, b), and T' the translation by (a', b'), then the succession of one after the other is the translation by $(a + a', b + b')$.
- Reflections:
- If the lines of the reflection are the same, we get the identity (that is, there is no motion: every point stays where it is.
- If the lines of the reflection are parallel, we get a translation.
- If the lines of the reflection intersect in a point, we get a rotation about that point.

Congruence

Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them. 8G2

Two figures are said to be *congruent* if there is a rigid motion that moves one onto the other. In high school mathematics the topic of congruence will be developed in a coherent, logical way, giving students the tools to answer many geometric questions. In 8th grade we are much more freewheeling, discovering what we can about congruence through experimentation with actual motions. In this section we will list some possible results that the class may discover; many classes will not discover some of these, but instead discover other interesting facts about congruence.

Example 12.

Using transparencies, decide, among the four triangles in Figure 15 , which are congruent.

Solution. Translating and then rotating appropriately, we can put triangle I on top of triangle II, so these are congruent. When we do the same, moving triangle I to triangle III, we find that the shortest leg of triangle I is shorter than the shortest leg of triangle II; since the short legs must correspond, these triangles are not congruent. Now, as for triangle IV, we can translate the right angle of triangle I to that of triangle IV. Since both short legs are vertical, we see they coincide. If we now reflect in that short leg, we land right on triangle IV. Thus, I and IV are also congruent.

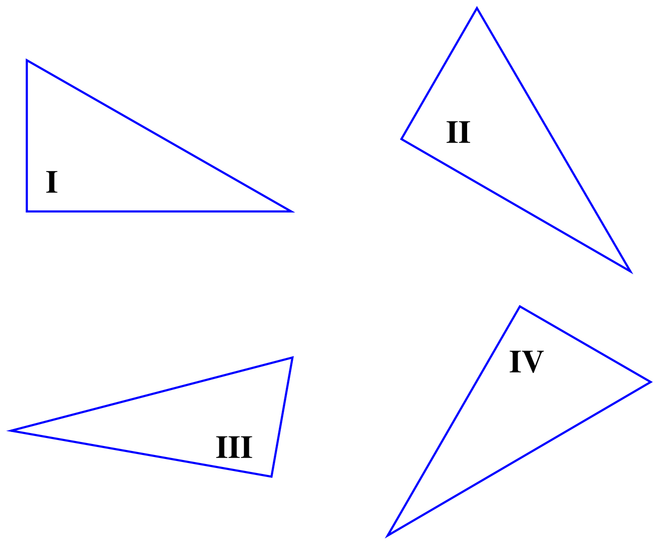


Figure 15

EXAMPLE 13.

All points in the plane are congruent.

Discussion. This may seem unnecessary to point out, but it does state a fact: given any points P and Q in the plane, there is a rigid motion taking P to Q. Actually, there are many. First of all, there is the translation of P to Q, and we can follow that by any rotation about the point Q. There is also a reflection of P to Q: fold the paper in such a way that P lands on top of Q. Then the crease line of the fold is the perpendicular bisector of the line segment PQ, and reflection in this line takes P to Q.

The following have been observed in the preceding section:

- Two line segments are congruent if they have the same length; otherwise they are not.
- Two angles are congruent if they have the same measure.
- Two rectangles are congruent if the side lengths of corresponding sides are the same. In particular, all squares of the same area are congruent, but not all rectangles of the same area are congruent.

Finding and demonstrating criteria for the congruence of triangles is a major topic in Secondary I; as an illustration, here we show how one such criterion, known as SSS, follows from the properties of rigid motions.

EXAMPLE 14.

Show SSS: Given two triangles, ABC and $A'B'C'$, if corresponding sides have the same length, then the triangles are congruent.

SOLUTION. What we have to find, given the two triangles, is a sequence of rigid motions that takes one onto the other. A good way to start is to try to create an example where this fails. Students will conclude that this is true, and find an explanation, in terms of the rigid motions desired. One such could be along the following lines. First of all, to make the argument more clear: let's designate the sides of ABC by the lower case version of the label of the opposite vertex: so, a is the side opposite vertex A , and so forth. Do the same for triangle $A'B'C$". Since line segments of the same length are congruent, we can move a onto a' by a rigid motion. Now draw a circle with center at C and radius of length b as in Figure 16. Since two circles intersect in at most two points, the circle of radius c centered at B intersects our circle either in the point A, or the point of reflection of A' in the line BC. Triangle $A'B'C$" has to be one of these triangles, one of which is the image of ABC. But the two triangles shown are congruent (by reflection in the line BC. So the original triangles are congruent.

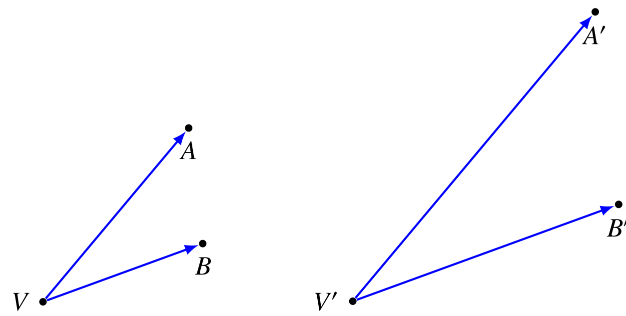


Figure 16

Section 9.2. Dilations and Similarity

Understand similarity in terms of rigid motions and dilations using ruler and compass, physical models, transparencies, geometric software. 8G3,4

Properties of dilations

Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates 8G3.

Verify that dilations take lines into lines, takes parallel lines to parallel lines and that a line and its image under a dilation are parallel.

Recall that in chapter 2, in the section on the slope of a line, we introduced the idea of a dilation in order to show that the slope of a line can be calculated using any two points. Let's start by reviewing that discussion.

A *dilation* is given by a point C, the *center* of the dilation, and a positive number r, the *factor* of the dilation. The dilation with center C and factor r moves each point P to a point P' on the ray CP so that the ratio of then length of image to the length of original is r: $|CP'|/|CP| = r$.

Note that if $r = 1$, nothing changes; this "'dilation" is called the *identity*. If $r > 1$, everything expands away from the center C, and if $r < 1$ everything contracts toward C.

The important fact about a dilation is that, for *every* line segment, the length of its image is r times the length of the segment. For students at this point, this will be easy to verify by experimentation, but the reason is not so obvious. In the next chapter we'll see this follows from the Pythagorean theorem; but it is also a fact of Euclidean geometry, as will be shown in the appendix.

Dilations are also directly connected to scale changes. Suppose that we start with an image of a rectangle, where the scale is in yards, as in figure 17(a). If we want more detail, we create a new image where the scale is in feet. Then, each interval (representing a yard) in the original image must be replicated across three intervals (each of which represents a foot) of the new image, and so our change of image is a scaling with scale factor 3. The result is figure (b). But, if instead we translate vertex A to vertex A' so (a) lands on the dashed rectangle in figure (b), then a dilation with center A' and factor 3 takes the small rectangle to the large one. Note that each length has been multiplied by 3, while the area of the larger rectangle is 9 times the area of the of the original rectangle. This will of course be true of every dilation of any figure: lengths are multiplied by the scale factor r, while areas are multiplied by the square of the scale factor, r^2.

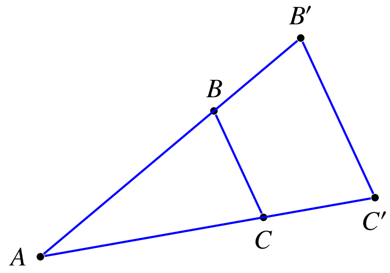


Figure 17A

Now we find the coordinate rule that expresses a dilation with center the origin. Suppose the factor of the dilation is r. Then the length of any interval is multiplied by r, so that the point $(x, 0)$ goes to the point $(rx, 0)$ and the point $(0, y)$ goes to the point $(0, ry)$. It follows (see figure 18) that $(x, y) \rightarrow (rx, ry)$.

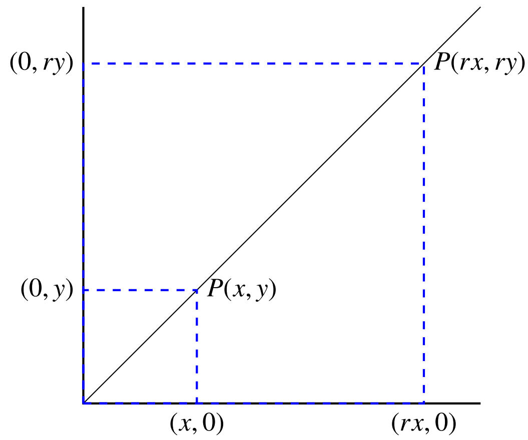


Figure 8

Properties of the dilation with center C and factor r:

- If P is moved to P', then $|CP'|/|CP| = r$.
- If P is moved to P' and Q is moved to Q', then $|Q'P'|/|QP| = r$.
- The dilation takes parallel lines to parallel lines.
- A line and its image are parallel.
- An angle and its image have the same measure.

All of these facts, except the last, were explored through examples in Chapter 2. Let's complete by showing that an angle and its image under a dilation are actually congruent. In figure 19, suppose that $\angle A'V'B'$ is obtained from $\angle AVB$ by a dilation. First of all, corresponding lines are parallel. Now, translate V to V'. Since corresponding lines under a translation are parallel, the ray VA must go to the ray $V'A'$ and the ray VB to the ray $V'B'$. Well, then the translation T takes the angle $\angle AVB$ to the angle $\angle A'V'B'$, so they have the same measure.

Similarity

Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them. 8G4

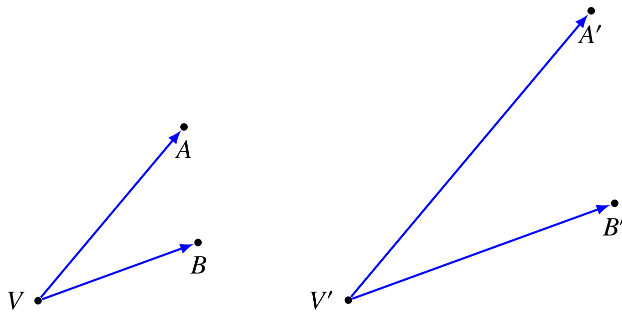


Figure 19

Two figures are said to be *similar* if there is a sequence of rigid motions and dilations that takes one figure onto the other. So, for example, the triangles in figure 14, the rectangles in figure 15 and the squares and triangles in figure 16 are similar in pairs, since there is a dilation with center the origin that places one on top of the other. Note that, if the dilation that places T onto T' has factor r, then the factor of the dilation placing T' onto T is r^{-1}. Let's list these facts about similarity:

Similar Figures

- Congruent figures are similar. This is because the "sequence of rigid motions and dilations" need not include any dilations, in which case it exhibits a congruence.
- Any two points or angles are similar; because they are congruent.
- Any two line segments or rays are similar. Let AB and CD both be either line segments or rays. First translate A to the point C, and then rotate the image of AB so that it and CD lie on the same line. First let's take the case of rays. Either the rays coincide, or they form the two different rays of the same line. In the second case, another rotation by 180° makes the rays coincide.

  Now let's look at two line segments. Move the segment AB, by rigid mottons as above, so that$A$ falls on C, and AB and CD now lie on the same ray starting from C. Let $r = |CD|/|AB|$. Then the dilation with center C and factor r places AB on top of CD.
- Any two circles are similar. Let C be the center of one of the circles and R its radius; and C' the center of the other, and R' its radius. Translate C to C'. Now the circles are concentric. Let $r = R'/R$. Then the dilation of factor r places the first circle on top of the other.
- For two similar triangles, the ratios of corresponding sides are all the same. This is because rigid motions do not change lengths, and dilations change all lengths by the same number, the factor of the dilation.
- For two similar triangles, the measure of corresponding angles is the same. This is because rigid motions and dilations do not change the measure of angles.

We end this discussion by showing that the the last statement is actually a criterion for similarity of triangles: if corresponding angles of triangles ABC and $A'B'C'$ have the same measure, the triangles are similar. We'll need another interesting fact for this:

Example 15.

If corresponding sides of triangles ABC and $A'B'C'$ are parallel, then the triangles are similar.

First, translate A to A'. Since a translation takes a line into a line parallel to it, the line of AB is moved to the line of $A'B'$ and the line of AC is moved to the line of $A'C'$, and the image of AB is parallel to $A'B'$, giving us the picture shown in Figure 20.

Now, for $r = |AB'|/|AB|$, the dilation with center A and factor r takes ABC onto $A'B'C'$.

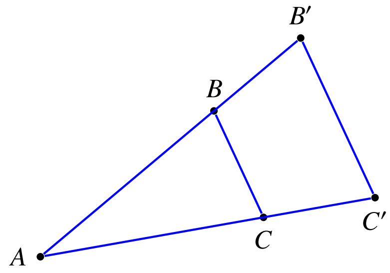


Figure 19

Example 16.

If corresponding angles of triangles ABC and $A'B'C'$ have the same measure, the triangles are similar.

Since any two angles of the same measure are congruent, we can find a sequence of rigid motions that takes $\angle CAB$ onto $\angle C'A'B'$. This puts us in the situation of figure 18, except that we do not know that the segments BC and $B'C'$ are parallel. But we do know (it is part of the hypothesis) that $\angle ABC$ and $\angle AB'C'$ are equal. That is enough; apply the dilation with center at A and factor $r = |AB'|/|AB|$. This takes B to B', and since the angle doesn't change, the ray BC lands on the ray $B'C'$. So the image of C under that dilation lies on the line of $B'C'$ and the line of AC', so it has to be C'. Thus the first triangle lands on top of the second triangle, and so they are similar,

Summary

Since the properties of the rigid motions and dilations have been gathered in the text, we shall just summarize the coordinate rules for the rigid motions and similarities that we have discussed.

- The coordinate rule for a translation by the vector (a, b) is $(x, y) \to (x + a, y + b)$.
- The coordinate rule for the reflection in the x-axis is $(x, y) \to (x, -y)$.
- The coordinate rule for the reflection in the y-axis is $(x, y) \to (-x, y)$.
- The coordinate rule for the reflection in the line $x = y$ is $(x, y) \to (y, x)$.
- The coordinate rule for the rotation by 90° is $(x, y) \to (y, x)$.
- The coordinate rule for the rotation by 180° is $(x, y) \to (-x, -y)$.
- The coordinate rule for the dilation with center the origin and factor r is $(x, y) \to (rx, ry)$.

Chapter 10
Geometry: Angles, Triangles and Distance

In section 1 we begin by gathering together facts about angles and triangles that have already been discussed in previous grades. This time the idea is to base student understanding of these facts on the transformational geometry introduced in the preceding chapter. As before, here the objective is to give students an informal and intuitive understanding of these facts about angles and triangles; all this material will be resumed in Secondary Mathematics in a more formal and logically consistent exposition.

Section 2 is about standard 8G6: *Explain a proof of the Pythagorean Theorem and its converse*. The language of this standard is very precise: it does not say *Prove...* but it says *Explain a proof of...*, suggesting that the point for students is to articulate their understanding of the theorem; not to demonstrate skill in reciting a formal proof. Although Mathematics tends to be quite rigorous in the construction of formal proofs, we know through experience, that informal, intuitive understanding of the "why" of a proof always precedes its articulation. Starting with this point of view, the student is guided through approaches to the Pythagorean theorem that make it believable, instead of formal arguments. In turn, the student should be better able to explain the reasoning behind the Pythagorean theorem, than to provide it in a form that exhibits form over grasp.

In Chapter 7, in the study of tilted squares, this text suggests that by replacing specific numbers by generic ones, we get the Pythagorean theorem. We start this section by turning this suggestion into an "explanation of a proof," and we give one other way of seeing that this is true. There are many; see, for example

jwilson.coe.uga.edu/EMT668/emt668.student.folders/HeadAngela/essay1/Pythagorean.html

The converse of the Pythagorean theorem states this: if $a^2 + b^2 = c^2$, where a, b, c are the lengths of the sides of a triangle, then the triangle is a right triangle, and the right angle is that opposite the side with length c. The Euclidean proof of this statement is an application of SSS for triangles. Although students played with SSS in Chapter 9, here we want a more intuitive and dynamic understanding. Our purpose here is to encourage thinking about dynamics, which becomes a central tool in later mathematics. We look at the collection of triangles with two side lengths a and b with $a \geq b$. As the angle at C grows from very tiny to very near a straight angle, the length of its opposing side steadily increases. It starts out very near $a - b$, and ends up very near $a + b$. There is only on triangle in this sequence where $a^2 + b^2$ is precisely c^2.

In the final section, we use the Pythagorean Theorem to calculate distances between points in a coordinate plane, This is what the relevant standard asks: it does not ask that students *know* the "distance formula." The goal here is that students understand the *process* to calculate distances: this process involves right angles and the Pythagorean theorem and students are to understand that involvement. Concentration on the formula perverts this objective.

Section 10.1 Angles and Triangles

Use informal arguments to understand basic facts about the angle sum and exterior angle o triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles. 8G5

In this section, we continue the theme of the preceding chapter: to achieve geometric intuition through exploration. We start with geometric facts that students learned in 7th grade or earlier, exploring them from the point of view of rigid motions and dilations.

(1). Vertical angles at the point of intersection of two lines have the same measure.

The meaning of *vertical* is in the sense of *a vertex*. Thus, in Figure 1, the angles at V with arrows are vertical angles, as is the pair at V without arrows.

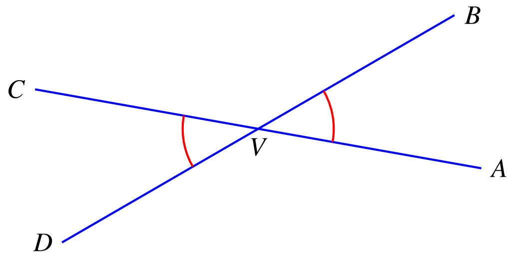


Figure 1

Now rotation with vertex V through a straight angle (180°) takes the line AC into itself. More specifically, it takes segment VA to VC and VB to VD, and so carries $\angle AVB$ to $\angle CVD$,. This rotation therefore shows that the angles $\angle AVB$ and $\angle CVD$ are congruent, and thusbhave the same measure.

The traditional argument (and that which appears in grade 7) is this: both angles $\angle AVB$ and $\angle CVD$ are supplementary to $\angle BVC$ (recall that two angles are *supplementary* if they add to a straight angle), and therefore must have equal measure. However, in grade 8 we want to understand measure equality in terms of congruence, and congruence in terms of rigid motions.

(2). If if two lines are parallel, and a third line L cuts across both, then corresponding angles at the points of intersection have the same measure.

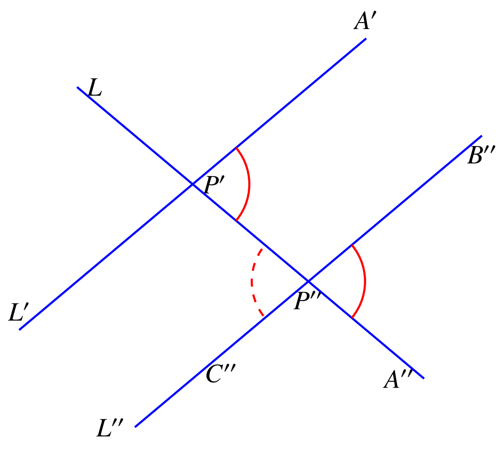


Figure 2

In Figure 2, the two parallel lines are L' and L'', and the corresponding angles are as marked at P' and P''. The translation that takes the point P' to the point P'' takes the line L' to the line L'' because a translation takes a line to another one parallel to it, and by the hypothesis, L'' is the line through P'' parallel to L'. Since translations also preserve the measure of angles, the corresponding angles as marked (at P' and P'') have the same measure. Now, because of (1) above, that opposing angles at a vertex are of equal measure, we can conclude that the angle denoted by the dashed arc is also equal to the angles denoted by the solid arc.

This figure also demonstrates the converse statement:

(3). Given two lines, if a third line L cuts across both so that corresponding angles are equal, then the two lines are parallel.

To show this, we again draw Figure 2, but now the hypothesis is that the marked angles at P' and P'' have the same measure. Since translations preserve the measure of angles, the translation of P' to P'' takes the angle $\angle A'P'P''$ to $\angle B''P''A''$, and so the image of L' has to contain the ray $P''B''$, and so is the line L''. Since the line L'' is the image of L' under a translation, these lines are parallel.

(4). The sum of the interior angles of a triangle is a straight angle.

In seventh grade, students saw this to be true by drawing an arbitrary triangle, cutting out the angles at the vertices, and putting them at the same vertex. Every replication of this experiment produces a straight angle. This experiment is convincing that the statement is true, but does not tell us why it is true.

We have two arguments to show why it is true. The first has the advantage that it uses a construction with which the student is familiar (that to find the area of a triangle) and thus reinforces that idea. The second has the advantage that it can be generalized to polygons with more sides. First, draw a triangle with a horizontal base (the triangle with solid sides in Figure 3). Rotate a copy of the triangle around the vertex B, and then translate the new triangle upwards to get the result shown in Figure 3, in which the triangle with dashed sides is the new position of the copied triangle. We have indicated the corresponding angles with the greek letters α, β, γ. Since the angles $\angle ABC$ and $\angle C'B'A$" have the same measure (β), the lines AB and $B'A'$ are parallel. Since they are parallel, the angles $B'A'C'$ and $\angle A'C'E$ have the same measure Now look at the point $B = C'$: the angles α, β, γ. form a straight angle.

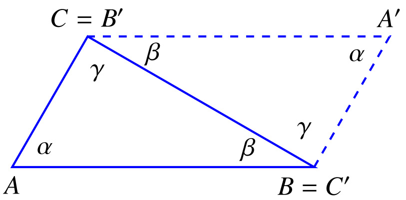


Figure 3

An alternative argument is based on Figure 4 below:

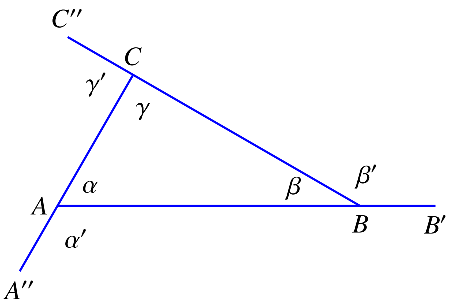


Figure 4

In this figure we have named the "exterior angles" of the triangle, α', β', γ', each of which is outside the triangle formed by the extension of the side of the triangle on the right. If we were to walk around the perimeter of the triangle, starting and ending at A looking in the direction of A', we would rotate our line of vision by a full circle, 360°. As this is the sum of the exterior angles, we have

$$\alpha' + \beta' + \gamma' = 360° .$$

But each angle in this expression is supplementary to the corresponding angle of the triangle, that is, the sum of the measures of the angle is 180°. So, the above equation becomes

$$(180° - \alpha) + (180° - \beta) + (180° - \gamma) = 360° ,$$

from which we get $\alpha + \beta + \gamma = 180°$.

We can generalize the second argument to polygons with more sides. Consider the quadrilateral in Figure 5.

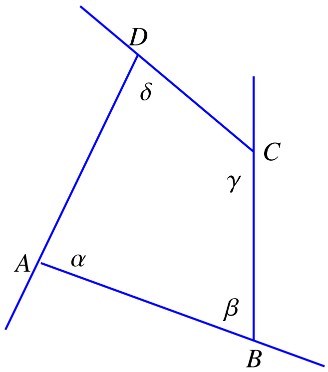


Figure 5

By the same reasoning as for the triangle, the sum of the exterior angles of the quadrilateral is also 360° and the sum of each interior angle and its exterior angle is 180°. But now there are four angles, so we end up with the equation

$$(180° - \alpha) + (180° - \beta) + (180° - \gamma) + 180° - \delta) = 360° , \quad \text{or} \quad 720° - (\alpha + \beta + \gamma + \delta) = 360° .$$

(5). The sum of the interior angles of a quadrilateral is 360°.

Can you now show that, for a five sided polygon, the sum of the interior angles of a quadrilateral is 540°? Can you go from there to the formula for a general polygon?

(6). If two triangles are similar, then the ratios of the lengths of corresponding sides is the same, and corresponding angles have the same measure.

(7). Given two triangles, if we can label the vertices so that corresponding angles have the same measure, then the triangles are similar.

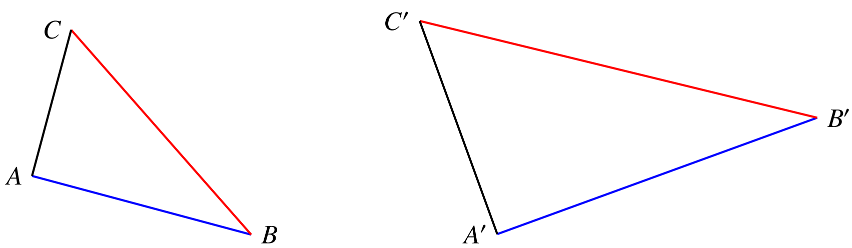


Figure 6a

We saw in Chapter 9 why (6) is true. Let us look more closely at statement (7).

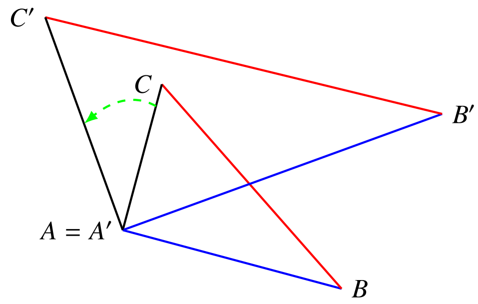


Figure 6b

Figure 6a shows a possible configuration of the two triangles. By a translation, we can place point A on top of pint A' to get Figure 6b. Now move the smaller triangle by a rotation with center $A = A'$, so that the point C lands on the segment $A'C'$. Since the angles $\angle CAB$ and $\angle C'A'B'$ have the same measure, the rotation must move line segment AB so that it lies on $A'B'$. Now, since $\angle ACB$ and $\angle A'C'B'$ have the same measure, the line segments CB and $C'B'$ must be parallel (by Proposition 2). Now the dilation with center A that puts point C on C', puts triangle ABC onto triangle $A'B'C'$, so they are similar.

The argument is not fully completed, for the configuration of Figure 6a is not the only possibility. In FIgure 6a, C is between A and B if we traverse the edge of the triangle in the clockwise direction, and the same is true for $\Delta A'B'C'$. However, this is not true in the configuration of Figure 7. Now how do you find the desired similarity transformation?

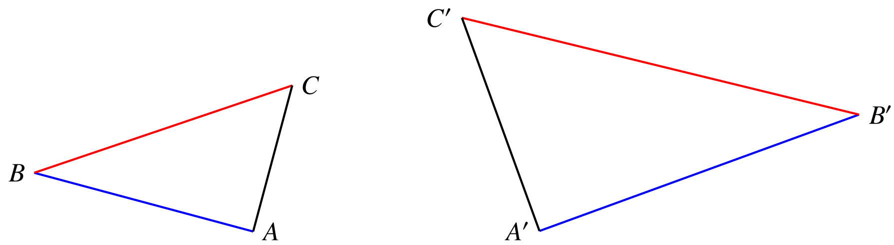


Figure 7

Again, we translate so that the points A and A' coincide. But now, the rotation that puts the line segment AB on the same line as $A'B'$ doesn't lead to the configuration of Figure 6b and the smaller triangle cannot be rotated so that corresponding sides lie on the same ray. But this is fixed by reflecting the smaller triangle in the line containing the segment AB and $A'B'$, and now we are in the configuration of Figure 6b. Indeed, we could have started with a reflection of the small triangle in the line through AC, and then followed the original argument.

Have we covered all cases? The answer is yes: the difference between Figure 6a and that of Figure 7 is that of orientation. So, if we start again with the two triangles ABC and $A'B'C'$with corresponding angles of the same measure, then we should first ask: is the orientation $A \to B \to C$ the same as the orientation $A' \to B' \to C'$ (both clockwise or both counter clockwise)? If so, we are in the case of Figure 6a. If not, after reflection in *any* side of triangle ABC puts us in the case of Figure 7.

Section 10.2 The Pythagorean Theorem.

Explain a proof of the Pythagorean Theorem and its converse. 8G6

In Chapter 7 we constructed "tilted" squares of side length, c whose area c^2 is a specific integer. If c is not an integer, we have observed that it cannot be expressed as a rational number (a quotient of integers). At the end of the discussion we mentioned that these specific examples generalized to a general theorem (known as the *Pythagorean* *theorem*) relating the lengths of the sides of a right triangle. This mathematical fact is named after a sixth century BCE mathematical society (presumed to be led by someone named *Pythagoras*). It is clear that this was known to much earlier civilizations: the written record shows it being used by the Egyptians for land measurements, and an ancient Chinese document even illustrates a proof. But the Pythagorean Society was given the credit for this by third century BCE Greek mathematicians. The Pythagorean Society is also credited with the discovery of constructible line segments whose length cannot be represented by a quotient of integers (in particular, the hypotenuse of a triangle whose legs are both the same integer.). The legend is that the discoverer of this fact was sacrificed by the Pythagoreans. We mention this only to highlight how much the approach to mathematics has changed in 2500 years; in particular this fact, considered "unfortunate" then, is now appreciated as a cornerstone of the attempt to fully understand the concept of number and its relationship to geometry.

Let's pick up with the discussion at the end of Chapter 7, section 1. There we placed our tilted squares in a coordinate plane so as to be able to more easily see the relationship between the areas of the squares and its associated triangles. Here, in order to stress that the understanding of the Pythagorean theorem does not involved coordinates, we look at those Chapter 7 arguments in a coordinate free plane. For two positive numbers, a and b, construct the square of side length $a + b$. This is the square bounded by solid lines in Figure 8, with the division points between the lengths a and b marked on each side. Draw the figure joining these points - this is the square with dashed sides in Figure 8. In Chapter 7 we observed that this is a square; now, let us check that this is so. First of all, triangle I is congruent to triangle II: translate triangle I horizontally so that the side labeled "a" lies on the side of triangle II labeled "b." Now rotate triangle I (in its new position) by 90°. Then the sides labeled a and b coincide and in the fact the triangles coincide, so are congruent. In the same way we can show that triange II is congruent to triangle III, and III is congrurent to IV.

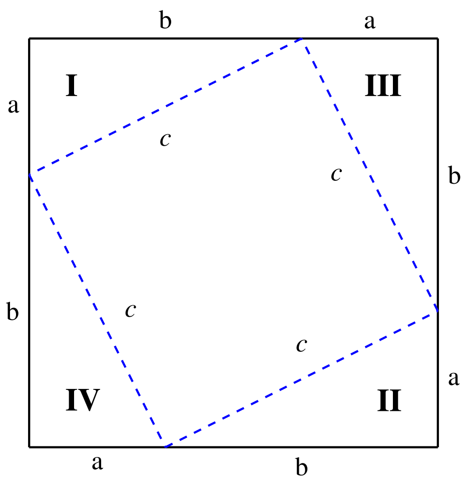


Figure 8

We conclude that the dashed lines are all of the same length, telling us that the dashed figure is a rhombus. To be a square, we have to show that one of the corner angles is a right angle. Lets concentrate on the vertex marked A, where the angle is marked γ. We see that

$$\alpha + \gamma + \beta = 180^\circ .$$

But the sum of the non-right trianges of a right triange is 90°, and these are, by the congruence of triangles I and IV, the same as the angles marked α and β; that is

$$\alpha + \beta = 90^\circ .$$

So, it follows that $\gamma = 90^\circ$.

In Chapter 7 we used these facts to calculate the area of the tilted square, which is c^2, where c is the length of the hypotenuse of the four triangles. By reconfiguring the picture as in Figure 9, we can show that this is also $a^2 + b^2$. This is known as the *Chinese proof* of the Pythagorean theorem, and records show that it precedes the Pythagorean Society by close to 1000 years.

Although this is a "proof without words," here is a description of how it goes: For the original square of side length $a + b$ is now subdivided in a different configuration: the bottom left corner is filled with a square of side length b, and the upper right corner, by a square of side length a.

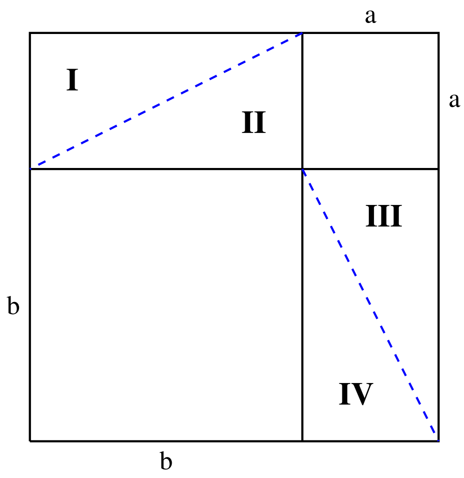


Figure 9

The rest of the big square of Figure 9 consists of four triangles, so what is outside those triangles has area $a^2 + b^2$. But each of those triangles is congruent to the triangle of the same label in Figure 8. Thus what is outside those four triangles must have area c^2. This result is:

The Pythagorean Theorem:

$$a^2 + b^2 = c^2$$

for a right triangle whose leg lengths are a and b and whose hypotenuse is of length c.

It is instructive to give another "proof without words" of the Pythagorean theorem; this one is due to Bhaskara, a 12th century CE mathematician in India. Start with the square of side length c (so of area c^2, and draw the right triangles of leg lengths a and b, with hypotenuse a side of this square, as shown in the top left of Figure 10. Now reconfigure these triangles and the interior square as in the image on the right. Since it is a reconfiguration, it still has area c^2. But now, by redrawing as in the lower image, we see that this figure consists of two squares, one of area a^2 and the other of area b^2.

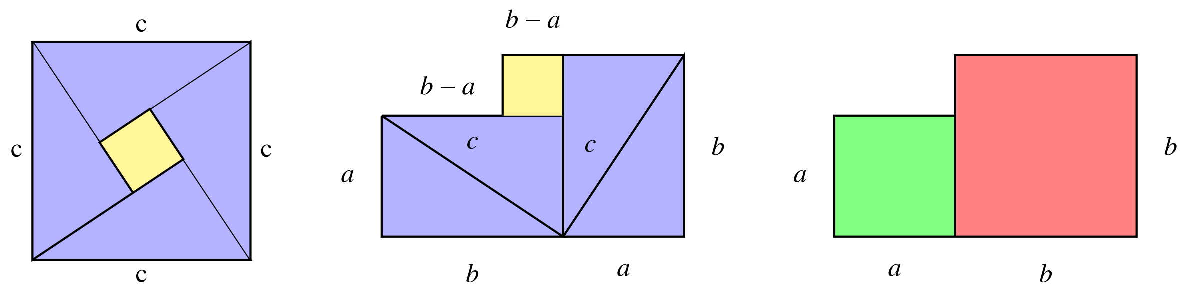


Figure 10

EXAMPLE 1.

a. A right triangle has leg lengths 6 in and 8 in. What is the length of its hypotenuse?

SOLUTION. Let c be the length of the hypotenuse. By the Pythagorean theorem, we know that

$$c^2 = 6^2 + 8^2 = 36 + 64 = 100 ,$$

so $c = 10$.

b. Another triangle has leg lengths 20 in and 25 in. Give an approximate value for the length of its hypotenuse.

SOLUTION.

$$c^2 = 20^2 + 25^2 = 400 + 625 = 1025 = 25 \times 41 .$$

So, $c = \sqrt{25 \times 41} = 5 \times \sqrt{41}$. Since $6^2 = 36$ and $7^2 = 49$, we know that $\sqrt{41}$ is between 6 and 7; probably a bit closer to 6. We calculate" $6.4^2 = 40.96$, so it makes good sense to use the value 6.4 to approximate $\sqrt{41}$. Then the corresponding approximate value of c is $5 \times 6.4 = 32$.

EXAMPLE 2.

a. The hypotenuse of a triangle is 25 ft, and one leg is 10 ft long. How long is the other leg?

SOLUTION. Let b be the length of the other leg. We know that $10^2 + b^2 = 25^2$, or $b^2 + 100 = 625$, and thus $b^2 = 525$. We can then write $b = \sqrt{525}$. If we want to approximate that, we first factor $525 = 25 \times 21$, sp $b = 5\sqrt{21}$ We can approximate $\sqrt{21}$ by 4.5 ($4.5^2 = 20.25$). and thus b is approximately given by $5 \times 4.5 = 22.5$.

b. An isosceles right triangle has a hypotenuse of length 100 cm. What is the leg length of the triangle.

SOLUTION. Referring to to the Pythagorean theorem, we are given: $a = b$ (the triangle is isosceles), and $c = 100$. So, we have to solve the equation $2a^2 = 100^2$. Since $100^2 = 10^4$, we have to solve $2a^2 = 10^4$, or $a^2 = 5 \times 10^3$. Write $5 \times 10^3 = 50 \times 10^2$, which brings us to $a = \sqrt{50 \times 10^2} = 10 \times \sqrt{50}$. Since $7^2 = 49$, we can give the approximate answers $a = 10 \times 7 = 70$.

EXAMPLE 3.

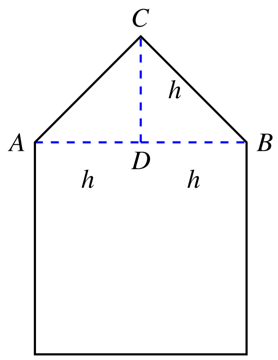


Figure 11

Figure 11 is that of an isosceles right triangle, ΔABC, lying on top of a square. The total area of the figure is 1250 sq. ft. What is the length of the altitude (CD) of the triangle? Note: This problem may be beyond the scope of 8th grade mathematics, but it still may be worth discussing, since it illustrates how the interaction of geometry and algebra works to solve a complex structural problem.

SOLUTION. Since the triangle is isosceles, the measure of $\angle CAD$ and $\angle CBD$ are both 45°. The altitude of a triangle is perpendicular to the base'; from which we conclude that the triangles ΔCAD and ΔCBD

are congruent. Since we want to find the length of CD, let's denote that number by h. This gives us the full labelling of Figure 11. The information given to us is about area, so let's do an area calculation. Since the length of a side of the square is $2h$, its area is $4h^2$. The area of the triangle on top of the square (one-half base times altitude) is $\frac{1}{2}(2h)(h) = h^2$. So, the area of the entire figure is $4h^2 + h^2 = 5h^2$, and we are given that that is 1250 sq. ft. We then have

$$5h^2 = 1250 \quad \text{so} \quad h^2 = 250 = 25 \times 10 \quad \text{and} \quad h = 5\sqrt{10}\,.$$

Now the Pythagorean theorem describes a relation among the lengths of the sides of a right triangle; it is also true that this relation describes a right triangle. This is what is called the *converse* of the Pythagorean theorem. The idea of "converse" is important in mathematics. Most theorems of mathematics are of the form : under certain conditions we must have a specific conclusion. The converse asks: if the conclusion is observed, does that mean that the given conditions hold, that is, does the equation $a^2 + b^2 = c^2$ relating the sides of a triangle tell us that the triangle s a right triangle?

Converse of the Pythagorean Theorem:

For a triangle with side lengths a, b, c if $a^2 + b^2 = c^2$, then $\angle ACB$ is a right angle.

In order to see why this is true, we show how to draw all triangles with two side lengths a and b. Let's suppose that $a \geq b$. On a horizontal line, draw a line segment BC of length a. Now draw the semicircle whose center is C and whose radius is b (see Figure 12). Then, any triangle with two side lengths a and b is congruent to a triangle with one side BC, and the other side the line segment from C to a point A on the circle.

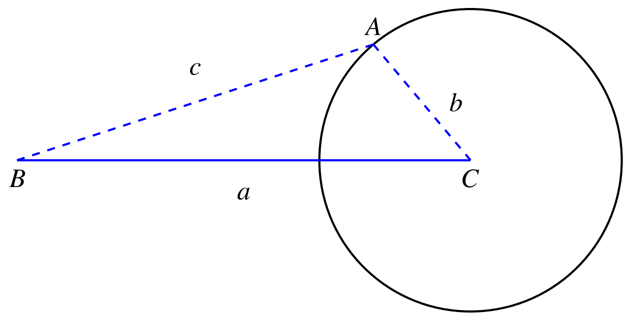


Figure 12

Now, as the line segment AC is rotated around the point C, the length c of the line segment BA continually increases. When AC is vertical, we have the right triangle, for which the length $c = \sqrt{a^2 + b^2}$. We conclude that for any triangle with side lengths a and b, the length c of the third side is either less than $\sqrt{a^2 + b^2}$ (triangle is acute), or greater than $\sqrt{a^2 + b^2}$ (triangle is obtuse) *except for the right triangle*(where the segment AB is vertical).

Example 4.

Draw a circle and its horizontal diameter (AB in Figure 13). Pick a point C on the circle. Verify by measurement that triangle ABC is a right triangle.

Solution. For the particular triangle the measures of the side lengths, up to nearest millimeter are: $AB = 44$ mm, $BC = 18$ mm, $AC = 40$ mm. Now, calculate: $BC^2 + AC^2 = 324 + 1600 = 1924$, and $AB^2 = 1936$. This is pretty close. If all students in the class get this close, all with different figures, then that is substantial statistical evidence for the claim that the triangle is always a right triangle.

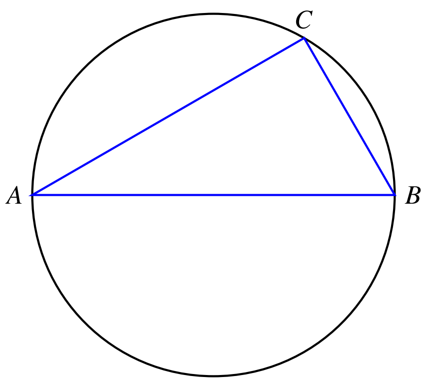


Figure 13

Section 10.3 Applications of the Pythagorean Theorem.

Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions. 8G7

Example 5.

What is the length of the diagonal of a rectangle of side lengths 1 inch and 4 inches?

Solution. The diagonal is the hypotenuse of a right triangle of side lengths 1 and 4, so is of length $\sqrt{1^2 + 4^2} = \sqrt{17}$.

Example 6.

Suppose we double the lengths of the legs of a right triangle. By what factor does the length of the diagonal change, and by what factor does the area change?

Solution. This situation is illustrated in Figure 14, where the triangles have been moved by rigid motions so that they have legs that are horizontal and vertical, and they have the vertex A in common. But now we can see that the dilation with center A that moves B to B' puts the smaller triangle on top of the larger one. The factor of this dilation s 2. Thus all length change by the factor 2, and area changes by the factor $2^2 = 4$.

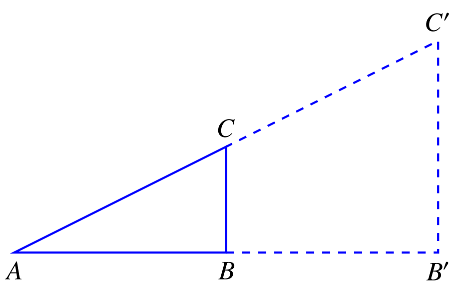


Figure 14

Example 7.

An 18 ft ladder is leaning against a wall, with the base of the ladder 8 feet away from the base of the wall Approximately how high up the wall is the top of the ladder?

Solution. The situation is visualized in Figure 15. The configuration is a right triangle with hypotenuse (the ladder) of length 18 feet, the base of length 8 feet, and the other leg of length h. By the Pythagorean theorem, we have

$$h^2 + 8^2 = 18^2 \quad \text{or} \quad h^2 + 64 = 324 .$$

Then $h^2 = 260$. Since we just want an approximate answer, we look for the integer whose square is close to 260: that would be 16 ($16^2 = 256$). So, the top of the ladder hits the wall about 16 feet above the ground.

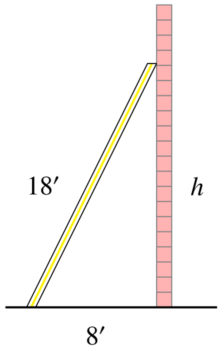
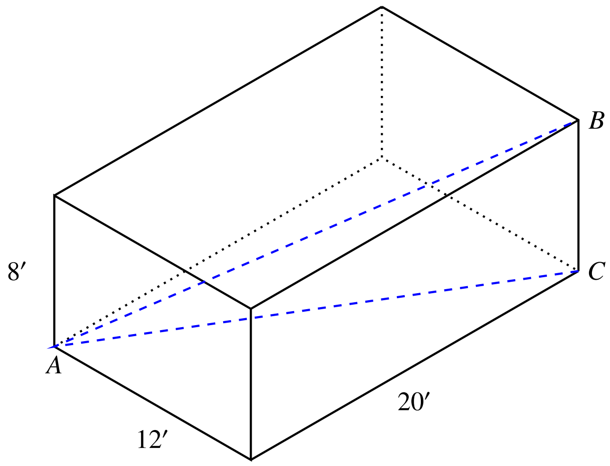


Figure 15

EXAMPLE 8.

A room is in the shape of a rectangle of width 12 feet, length 20 feet, and height 8 feet. What is the distance from one corner of the floor (point A in the figure) to the opposite corner on the ceiling?



Figure

SOLUTION. In Figure 16, we want to find the length of the dashed line from A to B. Now, the dash-dot line on the floor of the room is the hypotenuse of a right triangle of leg lengths 12 ft and 20 ft. So, its length is $\sqrt{12^2 + 20^2} = 23.3$. The length whose measure we want is the hypotenuse of a triangle ΔACB whose leg lengths are 23.3 and 8 feet. Using the Pythagorean theorem again we conclude that the measure of the line segment in which we are interested (AB) is $\sqrt{8^2 + 23.3^2} = 24.64$; since our original data were given in feet, the answer: 25 ft. should suffice.

EXAMPLE 9.

What is the length of the longest line segment in the unit cube?

SOLUTION. We can use the same figure as in the preceding problem, taking that to be the unit cube.

Then the length of the diagonal on the bottom face is $\sqrt{1^2 + 1^2} = \sqrt{2}$ units, and the length of the diagonal AB is $\sqrt{1^2 + (\sqrt{2})^2} = \sqrt{3}$.

These examples show that we can use the Pythagorean theorem to find lengths of line segments in space. Given the points A and B, draw the rectangular prism with sides parallel to the coordinate planes that has A and B as diametrically opposite vertices (refer to Figure 16). Then, as in example 8, the distance between A and B is the square root of the sum of the lengths of the sides.

Example 10.

What is the length of the longest line segment in a box of width 10", length 16" and height 8"?

$$\text{Length} = \sqrt{10^2 + 16^2 + 8^2} = \sqrt{100 + 256 + 64} = \sqrt{428} = \sqrt{4 \cdot 107} = 2\sqrt{107}$$

inches, or a little more that 20 inches.

Example 11.

In the movie *Despicable Me*, an inflatable model of The Great Pyramid of Giza in Egypt was created by Vector to trick people into thinking that the actual pyramid had not been stolen. When inflated, the false Great Pyramid had a square base of side length 100 m. and the height of one of the side triangles was 230 m. What is the volume of gas that was used to fully inflate the fake Pyramid?

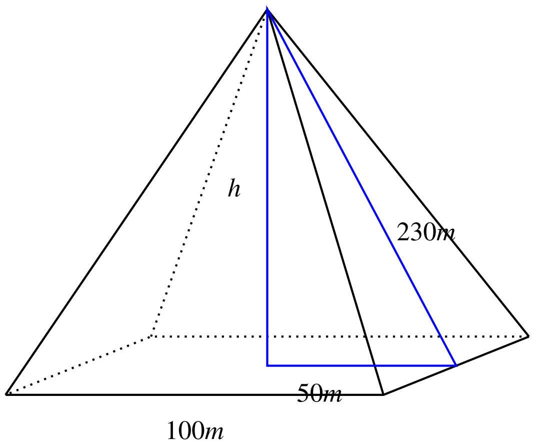


Figure 17

Solution. The situation is depicted in Figure 17. Now, we know that the formula for the volume of a pyramid is $\frac{1}{3}Bh$, whee B is the area of the base and h is the height of the pyramid (the distance from the base to the apex, denoted by h in the figure). Since the base is a square of side length 100 m., its area is 10^4 m^2. To calculate the height, we observe (since the apex of the pyramid is directly above the center of the base), that h is a leg of a right triangle whose other leg is 50 m. and whose hypotenuse has length 230 m. By the Pythagorean theorem $h^2 + 50^2 = 230^2$. Calculating we find $h^2 = 230^2 - 50^2 = 52900 - 2500 = 50400$. Taking square roots, we have $h = 225$ approximately. Then, the volume of the pyramid is

$$\text{Volume} = \frac{1}{3}(10^4)(2.25 \times 10^2) = .75 \times 10^5 = 75,000 \text{ m}^3.$$

Section 10.4 The Distance Between Two Points

Apply the Pythagorean theorem to find the distance between two points. 8G8.

For any two points P and Q , the *distance* between P and Q is the length of the line segment PQ.

We can approximate the distance between two points by measuring with a ruler, and if we are looking at a scale drawing, we will have to use the scale conversion. If the two points are on a coordinate plane, we can find the distance between the points using the coordinates by applying the Pythagorean theorem. The following sequence of examples demonstrates this method, starting with straight measurement.

EXAMPLE 12.

a. Using a ruler, estimate the distance between each of the three points P, Q and R on Figure 18.

SOLUTION. The measurements I get are PQ = 39 mm; PR = 39 mm and QR = 41 mm. Of course, the actual measures one gets will depend upon the display of the figure.

b. Now measure the horizontal and vertical line segments (the dashed line segments in the figure), to confirm the Pythagorean theorem.

SOLUTION. The horizontal line is 30 mm, and the vertical line is 27 mm. Now $30^2 + 27^2 = 900 + 729 = 1629$, and $39^2 = 1621$. This approximate values are close enough to confirm the distance calculation, and attribute the discrepancy to minor imprecision in measurement.

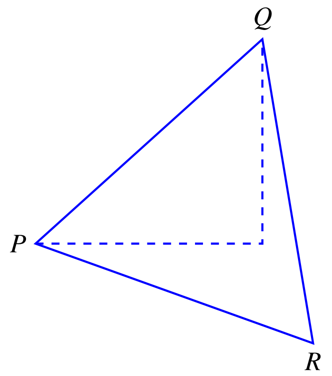


Figure 18

EXAMPLE 13.

In the accompanying map of the northeast United States, one inch represents 100 mies. Using a ruler and the scale on the map, calculate the distance between

a. Pittsburgh and Providence

b. Providence and Concord

c. Pittsburgh and Concord

d. Now test whether or not the directions Providence → Pittsburgh and Providence → Concord are at right angles.

SOLUTION. After printing out the map, we measured the distances with a ruler, and found

a. Pittsburgh to Providence: Four and an eighth inches, or 412.5 miles;

b. Providence to Concord: 15/16 of an inch, or 93.75 miles;

US: Northeast Region

c. Pittsburg to Providence: Four and a quarter inches, or 425 miles.

d. Since we are only approximating these distances, we round to integer values and then check the Pythagorean formula to see how close we come to have both sides equal Let us denote these distances by the corresponding letters a, b, c, so that $a = 413$, $b = 94$, $c = 425$. We now calculate the components of the Pythagorean formula:

$$a^2 = 170,569,\ b^2 = 8,836,\ \text{so}\ a^2 + b^2 = 179,405\ ; \quad c^2 = 180,625\ .$$

The error, 1220, is well within one percent of c^2, so this angle can be taken to be a right angle.

Example 14.

On a coordinate plane, locate the points $P(3,2)$ and $Q(7,5)$ and estimate the distance between P and Q. Now draw the horizontal line starting at P and the vertical line starting at Q and let R be the point of intersection. Calculate the length of PQ using the Pythagorean theorem.

Solution. First of all, we know the coordinates for R: (7,2). So the length of PR is 4, and the length of QR is 3. By the Pythagorean theorem, the length of PQ is $\sqrt{4^2 + 3^2} = 5$. The measurement with ruler should confirm that.

Example 15.

Find the distance between each pair of these three points on the coordinate plane: $P(-3,2)$, $Q(7,7)$ and $R(2,-4)$.

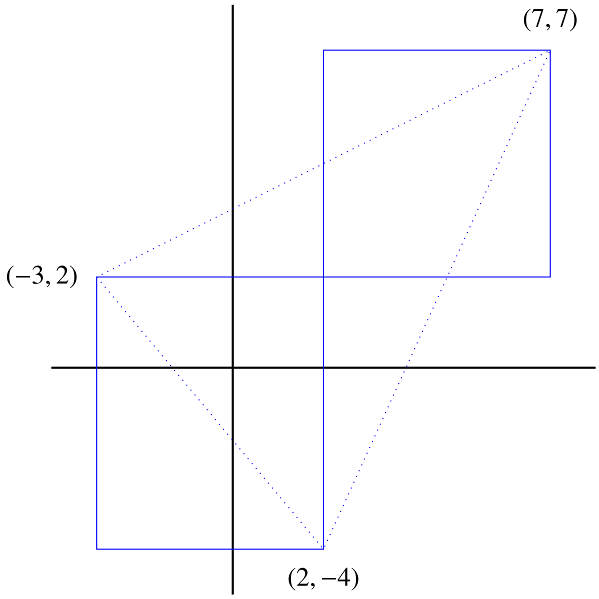


Figure 19

SOLUTION. In Figure 19 have drawn the points and represented the line joining them by dotted lines. To calculate the lengths of these line segments, we consider the right triangle with horizontal and vertical legs and PQ as hypotenuse. The length of the horizontal leg is 7-(-3) = 10, and that of the vertical leg is 7-2 = 5. So

$$|PQ| = \sqrt{10^2 + 5^2} = \sqrt{5^2(2^2 + 1)} = 5\sqrt{5} .$$

For the other two lengths, use the slope triangles as shown and perform the same calculation:

$$|QR| = \sqrt{11^2 + 5^2} = \sqrt{121 + 25} = \sqrt{146} ,$$

$$|PR| = \sqrt{5^2 + 6^2} = \sqrt{61} .$$

These examples show us that the distance between two points in the plane can be calculated using the Pythagorean theorem, since the slope triangle with hypotenuse the line segment joining the two points is a right angle. This can be stated as a formula, using symbols for the coordinates of the two points, but it is best if students understand the protocol and the reasoning behind it, and by no means should memorize the formula. Nevertheless, for completeness, here it is.

The Coordinate Distance Formula:

Given points $P : (x_0, y_0)$, $Q(x_1, y_1)$

$$|PQ| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} .$$

EXAMPLE 16.

Figure 20 is a photograph, by Jan-Pieter Nap, of the Mount Bromo volcano on the island Java of Indonesia taken on July 11 2004.

http://commons.wikimedia.org/wiki/File:Mahameru-volcano.jpeg

Figure 20

From the bottom of the volcano (in our line of vision) to the top is 6000 feet. Given that information, by measuring with a rule, find out how long the visible part of the left slope is, and how high the plume of smoke is.

Solution. Using a ruler, we find that the height of the image of the volcano is 12 mm, the length of the left slope is 21 mm, and the plume is 10 mm high. Now we are given the information that the visible part of the volcano is 5000 feet, and that is represented in the image by 12 mm. Thus the scale of this photo (at the volcano) is 12 mm : 6000 feet, or 1 mm : 500 feet. Then the slope is $21 \times 500 = 10,500$ feet, and the plume is $10 \times 500 = 5000$ feet high.

Example 17.

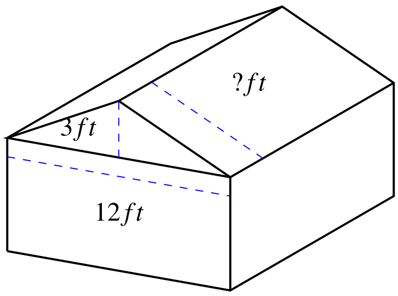


Figure 21

In my backyard I plan to build a rectangular shed that is 12' by 20', with a peaked roof, as shown in Figure 21. The peak of the roof is 3' above the ceiling of the shed. How long do I have to cut the roof beams?

Example 18.

Figure 22 is a detailed map of part of the Highline Trail, courtesy of www.christine@lustik.com.

Using the scale, find the distance from Kings Peak to Deadhorse Pass as the crow flies. Now find the length of the trail between these points. In both cases, just measure distances along horizontal and vertical lens and use the Pythagorean theorem. Measuring with a ruler on the scale at the bottom, we

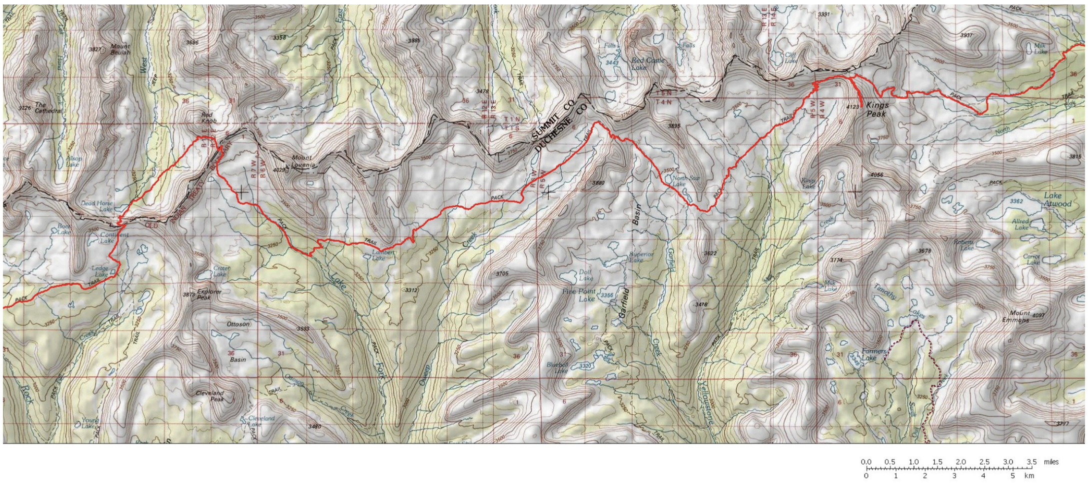


Figure 22

find that the scale is 20 mm : 5 km, or 4 mm/km. Measuring the distance from King's Peak to Deadhorse Pass, we get 108 mm. Then the actual distance in km is

$$(108 \text{ mm}) \cdot \frac{1 \text{ km}}{4 \text{ mm}} = 27 \text{ km} .$$

Now, to find the length of the trail, you will have to measure each straight length individually and add the measurements. Alternatively, you can go to http://lustik.com/highline_trail.htm and read an entertaining account of the hike.